Supplementary Material for: The Jamming Energy Landscape is Hierarchical and Ultrametric

R. C. Dennis and E. I. Corwin
Department of Physics and Materials Science Institute,
University of Oregon, Eugene, Oregon 97403, USA.

THE SUBDOMINANT ULTRAMETRIC

We present here a simple outline of the algorithm for creating the subdominant ultrametric and provide an intuitive explanation for how it works. Given a metric, d, the corresponding subdominant ultrametric, $d^{<}$, can be found with the following algorithm [1]:

1. The metric, d, is a symmetric matrix of pairwise distances between minima. This can be reinterpreted as an edge-weighted graph where the nodes are the minima and the edge weights are the distances between minima.

2. We compute the minimum spanning tree of this graph, which is simply the network with the minimum possible total edge weight (sum of distance values) which connects every node into a single tree. The minimum spanning tree is unique [2] and has the property that every pair of nodes has only one path connecting them.

3. The subdominant ultrametric, $d^{<}$, is created as a symmetric matrix with entries $d^{<}_{ij}$ determined by the maximum edge weight in the path from node i to node j in the minimum spanning tree.

The hierarchical nature of $d^{<}$ derives from that of the minimum spanning tree. The maximum condition ensures that the resulting metric will be ultrametric as it is a direct enforcement of the ultrametric inequality (Equation 1 of the text). Given any triplet of minima the constructed subdominant ultrametric $d^{<}$ will produce a triangle with one short side and two equal long sides. This is an equivalent definition of the ultrametric inequality given in the manuscript. This explanation demonstrates why this algorithm returns an ultrametric which always contains distances that are less than or equal to the corresponding metric entries. However, the proof that this is the largest possible ultrametric to satisfy this criterion is less intuitively obvious and can be found in the original reference [1].

Uniqueness of the Subdominant Ultrametric

If all of the edge-weights in our metric are distinct, the minimum spanning tree will be unique [2]. Our metrics come from amorphous systems with unique edge-weights and unique minimum spanning trees. Additionally, the subdominant ultrametric is always unique whether or not this condition is met because while degenerate edge-weights results in multiple minimum spanning trees, the maximum edge-weight along the path between every pair of nodes will be the same [1].

[1] R. Rammal, J.C. Angles d'Auriac, and B. Doucot, "On the degree of ultrametricity," Journal de Physique Lettres **46**, 945–952 (1985).
[2] Joseph B. Kruskal, "On the shortest spanning subtree of a graph and the traveling salesman problem," Proc. Amer. Math. Soc. **7**, 48–50 (1956).