



Supplementary Information for

A direct link between active matter and sheared granular systems

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This PDF file includes:

Supplementary text
Figs. S1 to S3
SI References

Supporting Information Text

1. Generating random vectors for AQRD

A. Wave-like correlated fields (WCF). We generate the vector $|c\rangle$ under the constraint that it must be continuous across the boundary, and thus periodic. In order to generalize AQS strain, we set the y -component of each \mathbf{c}_i to zero and let the x -component depend on the height of the particle

$$\mathbf{c}_i = \sin\left(\frac{\pi y_i}{\xi}\right) \hat{x} \quad [\text{S1}]$$

where \hat{x} is the unit vector in the x -direction and ξ is the correlation length. In order for this to be periodic, we must have $\xi = L_y/n$ with $n \in \mathbb{Z}$. We note that AQS with Lees-Edwards boundary conditions is simply $\xi = L_y$ with a phase shift (or simply a cosine). Once $|c\rangle$ is determined, it is normalized so that $\langle c|c\rangle = 1$.

B. Gaussian correlated fields (GCF). This section describes the process of generating a random Gaussian vector $|c\rangle$ with finite spatial correlation length ξ using Fourier transforms. This method is used for all GCF ensembles, except $\xi = 1$ (corresponding to completely uncorrelated field) where each component of $|c\rangle$ is drawn from a uniform distribution and then normalized to length 1.

The two-dimensional system is a box of size $L_x \times L_y$ with periodic boundary conditions, allowing us to define wave vectors $\mathbf{k}_{nm} = \left(\frac{a\pi n}{L_x}, \frac{b\pi m}{L_y}\right)$ for $n, m \in \mathbb{Z}$ where $a = \pm 2$ and $b = \pm 2$, with signs decided randomly, as discussed below. In practice, we truncate the Fourier sums at $n, m = Q$ taking $Q = 20$.

To create $|c\rangle$, we need a correlated random field $\Psi(\mathbf{x})$ which is Gaussian distributed with zero mean $\langle \Psi(\mathbf{x}) \rangle = 0$ and has the two-point correlator $\langle \Psi(\mathbf{x})\Psi(\mathbf{x}') \rangle = f(|\mathbf{x} - \mathbf{x}'|)$. We enforce $f(x)$ to be a Gaussian function, whose explicit Fourier transform is $\tilde{f}(|\mathbf{k}|) = \exp\left[-\frac{|\mathbf{k}|^2 \xi^2}{(a^2 + b^2)}\right]$.

First, we generate a set of uncorrelated random fields $\tilde{\psi}(\mathbf{k})$ with $\langle \tilde{\psi}(\mathbf{k}) \rangle = 0$ and $\langle \tilde{\psi}(\mathbf{k})\tilde{\psi}(\mathbf{k}') \rangle = \frac{1}{4\pi^2} \delta_{\mathbf{k}, -\mathbf{k}'}$, where δ is the Kronecker function, and the factor $4\pi^2$ comes from the Fourier transform convention. In practice, for each wave vector of the truncated sum we generate a random field $\tilde{\psi}(\mathbf{k}_{nm}) = A(\mathbf{k}) \exp\{iB(\mathbf{k})\}$, with $A(\mathbf{k}) = A(-\mathbf{k})$ normally distributed with zero mean and variance $\frac{1}{4\pi^2}$, and $B(\mathbf{k}) = -B(-\mathbf{k})$ uniformly distributed on the interval $[0, 2\pi]$.

Secondly, we use the Fourier transform of the target correlator $f(x)$ to construct a new field $\tilde{\Psi}(\mathbf{k}) = \tilde{f}(|\mathbf{k}|) \tilde{\psi}(\mathbf{k})$, whose Fourier transform of the α component is

$$\Psi^\alpha(\mathbf{x}) = \sum_{n,m=1}^Q A_{nm}^\alpha e^{-|\mathbf{k}_{nm}|^2 \xi^2 / (a^2 + b^2)} \cos(B_{nm}^\alpha + \mathbf{k}_{nm} \cdot \mathbf{x}). \quad [\text{S2}]$$

The random vectors $\mathbf{c}_i = (c_i^x, c_i^y)$ are then defined as $c_i^\alpha = \Psi^\alpha(\mathbf{x}_i)$ using the initial positions \mathbf{x}_i . Once $|c\rangle$ is determined, it is normalized so that $\langle c|c\rangle = 1$.

It is important to note that because these fields are built using Fourier transforms, they will have a bulk phase preference at high values of ξ – greater than about a quarter of the box size – along the $\tan^{-1}\left(\frac{b}{a}\right)$ axis. In other words, the multiplicity of fields that obey the necessary conditions – that they have zero mean and they respect the periodic boundary conditions – becomes very small as ξ approaches the box size, and in fact in the limit that ξ equals the box size, only the simple shear strain field satisfies those conditions.

Therefore, for GCF fields we restrict ourselves to values of ξ for which this bulk phase preference does not dominate. It is also for this reason that the signs of a and b are chosen at random for each instantiation of a system. Other even integer values of a and b may be used to create Gaussian correlated fields, but these fundamentally alter the symmetry of the system for high values of ξ , and thus are not used in this work.

2. Athermal quasistatic random forcing (AQRF)

In addition to the strain-controlled definition of AQRD in the main text, we can also consider a stress-controlled version denoted athermal quasistatic random forcing (AQRF). Instead of enforcing a displacement vector $|c\rangle$, we apply an external force $|F^{ext}\rangle = f|c\rangle$ on the system and we measure the resulting strain, where again $\langle c|c\rangle = 1$. Thus, instead of a constrained minimization, we simply perform a minimization subject to a fixed external force. Once minimized, the sum of all forces on each particle must be zero:

$$\mathbf{F}_i = \sum_{j \in \partial i} \mathbf{F}_{ij} + f\mathbf{c}_i = 0, \quad [\text{S3}]$$

where $j \in \partial i$ indicates a sum over interparticle forces acting on particle i . The sum of interparticle forces is parallel to $|c\rangle$, allowing closure with $|c\rangle$ to give $f = -\langle c|F\rangle$. Following the arguments around Eq. (2) a normalization factor is necessary to compare the size of stresses in AQS leading to

$$\tilde{\sigma} = -\frac{f}{L_x} \sqrt{\frac{N}{12}}. \quad [\text{S4}]$$

The random strain is then naturally defined as

$$\tilde{\gamma} = \frac{1}{A} \frac{dU}{d\tilde{\sigma}} = \frac{1}{L_x L_y} \sum_{i=1}^N \frac{\partial U}{\partial \mathbf{x}_i} \cdot \frac{d\mathbf{x}_i}{d\tilde{\sigma}}. \quad [\text{S5}]$$

where Eq. (S3) ensures that $\frac{\partial U}{\partial \mathbf{x}_i} = \sum_{j \in \partial i} \mathbf{F}_{ij} = -f \mathbf{c}_i$, and $\frac{d\mathbf{x}_i}{d\tilde{\sigma}}$ can be interpreted for small stresses as the displacement $\Delta \mathbf{x}_i$ of particle i , subject to the applied stress $\tilde{\sigma}$, leading to the natural relation

$$\frac{d\mathbf{x}_i}{d\tilde{\sigma}} = -\frac{\Delta \mathbf{x}_i L_x}{f \sqrt{\frac{N}{12}}}. \quad [\text{S6}]$$

Together these lead to the definition of random strain in the stress-controlled ensemble:

$$\tilde{\gamma} = \frac{\langle c|\Delta x \rangle}{L_y \sqrt{\frac{N}{12}}} \quad [\text{S7}]$$

Physically, this scalar product is simply the sum over all particles of their individual displacements along their corresponding imposed direction \mathbf{c}_i . This is the natural counterpart of the random stress definition given in Eq. (5).

3. Generality of the stress-strain collapse

In Fig. 5d-f, we show the collapse of the average stress-strain curves, the probability distribution of stress drops, and the probability distribution of the strain intervals between rearrangements. Here, we show those same curves for both WCF fields (Fig. S1) and for GCF systems of different size at constant correlation length (Fig. S2). Empirically, Fig. S2b shows that the distribution of stress drops and the distribution of strain intervals between rearrangements collapse when scaled as $\sqrt{N} \Delta \tilde{\sigma} / \sqrt{\kappa}$ and $\sqrt{N} \tilde{\gamma} \sqrt{\kappa}$, in contrast to the geometric mean values $\langle \tilde{\sigma} \rangle \sim p/N$ and $\langle \tilde{\gamma} \rangle \sim p^{1/3}/N$. This seeming discrepancy is due to the choice of a fixed strain step size which systematically under-counts any small rearrangements. The full distribution appears to be a power law distribution with a minimum stress drop known to scale as $1/N$ (see supplement of Ref. (1)). Thus, instead of fitting the center of the distribution, we ignore the small cutoff and match the high tail.

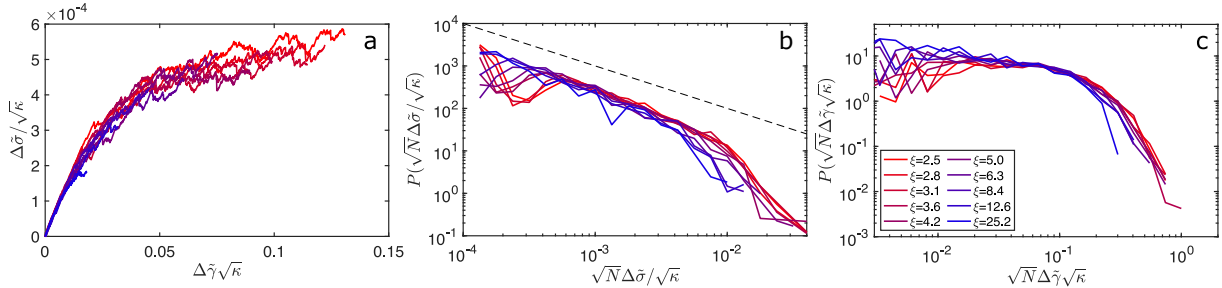


Fig. S1. The same scaling which in Fig. 5 collapses the AQRD data for Gaussian Correlated Fields is applied to Wave-like Correlated Fields (WCF) of varying correlation length ξ . Here the **a)** average stress-strain curves, **b)** distribution of the stress drops, and **c)** the distribution of strain intervals between events all collapse when scaled with the ratio of initial shear moduli $\tilde{\mu}_0$ vs. the AQS initial shear modulus μ_0 as $\kappa = \tilde{\mu}_0 / \mu_0$. The dotted line in panel b is of slope -1 . All systems have $N = 2048$ particles.

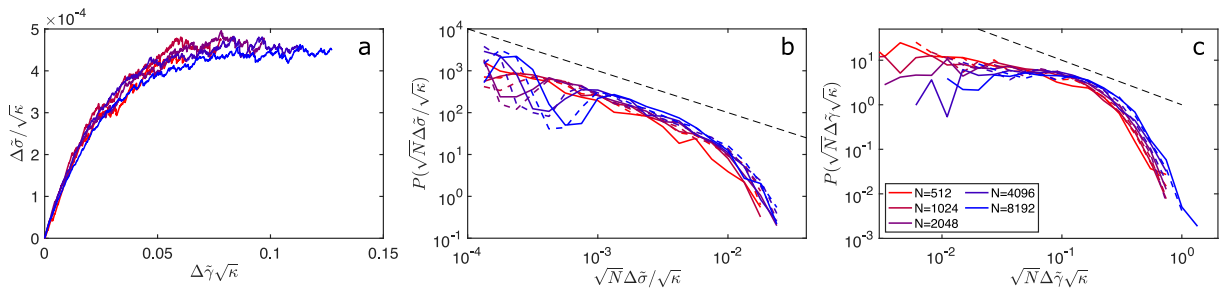


Fig. S2. The same scaling which in Fig. 5 collapses the AQRD data for Gaussian Correlated Fields with $N = 2048$ particles is applied to systems of varying size. Here the **a)** average stress-strain curves, **b)** distribution of the stress drops, and **c)** the distribution of strain intervals between events all collapse when scaled with the ratio of initial shear moduli $\tilde{\mu}_0$ vs. the AQS initial shear modulus μ_0 as $\kappa = \tilde{\mu}_0 / \mu_0$. Dashed lines in b and c represent AQS systems of the same size. The dotted line in panel b is of slope -1 . All systems use GCF fields with $\xi = 1$. While curves are generated with the same protocol detailed in the text, it is clear that the dynamic range of the strain should be applied to a maximum value which is proportional to \sqrt{N} .

4. Scaling argument supporting our mean-field prediction in finite dimension

In order to gain more physical intuition on our infinite-dimensional mean-field results, we present thereafter a scaling argument at a finite dimension d which supports the mean-field prediction that the total effective strain would be given by $\gamma_{\text{eff}} = \tilde{\gamma} \sqrt{\mathfrak{F}}/\ell$ and that the typical elastic modulus should scale as $\mu \sim \mathfrak{F}/\ell^2$. The different assumptions underlying this argument turn out to be exact in infinite-dimension, where we were able to obtain this prediction from the computations, even before having any physical intuition of what to expect.

Let's first consider the most extreme case with an infinite ξ . This means that all particles are driven with a same vector \mathbf{c}_i : the relative local strains are all strictly zero, with no variance ($\mathfrak{F} = 0$) or equivalently a distribution of relative strains $\bar{\mathcal{P}}(\mathbf{c}_{ij}) = \delta(\mathbf{c}_{ij})$. The whole system is thus simply translated in space, and its effective strain always remains strictly zero.

If instead we allow for a large but finite ξ , the system can be pictured as being composed of large 'patches' in which particles share the same vector \mathbf{c}_i , and pairs have thus zero relative strains \mathbf{c}_{ij} inside a given patch. Only pairs of particles living at the boundaries between such patches experience a non-zero \mathbf{c}_{ij} , and can thus contribute to the total effective strain felt by the system under AQRD. The smaller ξ , the larger the proportion of pairs at a boundary. We can quantify this within the simplified patchy picture: let's assume that we alternate in 2d, as in a chessboard, patches with $\mathbf{c}_i = \pm \mathbf{A}$. Inside a patch $\mathbf{c}_{ij} = 0$; pairs at the boundaries have $\mathbf{c}_{ij} = \pm 2\mathbf{A}$, as illustrated in Fig. S3. We can estimate the proportion of interacting pairs living at a boundary as $\rho = L/\xi \times \ell \times L \times 2/L^2 = 2\ell/\xi$. In dimension d this generalizes to $\rho = d\ell/\xi$. The variance of relative strains is then given by

$$\mathfrak{F}/\ell^2 = \sum_{\text{interacting pairs}} \mathbf{c}_{ij}^2 = \sum_{\text{in a patch}} (1-\rho) \times 0^2 + \sum_{\text{in a patch}} \rho \times (2\mathbf{A})^2 = 4A^2 d\ell/\xi.$$

Consequently, because the relative strain on interacting pairs is distributed, the typical strain applied to a given pair scales as $\sim \tilde{\gamma} \sqrt{\mathfrak{F}}/\ell = \tilde{\gamma} 2A\sqrt{d\ell/\xi}$.

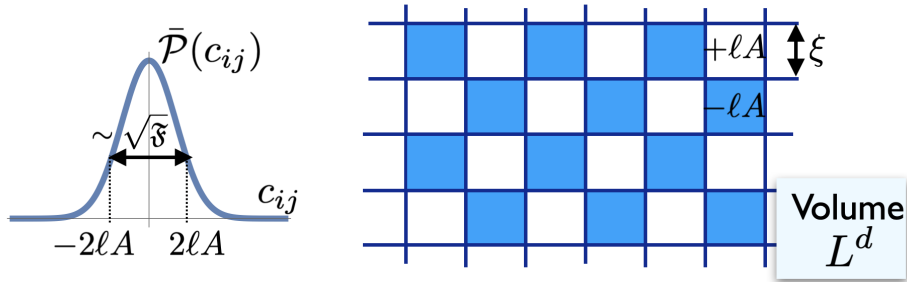


Fig. S3. *Left:* If we have a Gaussian distribution of relative strains of zero mean and variance \mathfrak{F} , we can simplify it in a patchy picture where we retain only the value at the peak ($\mathbf{c}_{ij} = 0$) or its standard deviation ($\mathbf{c}_{ij} = \pm 2\ell A$). *Right:* The associated patchy representation of the displacement field itself, where $\mathbf{c}_i = \pm \ell A$ on alternating patches.

Beyond this simplified patchy picture, Eq. (6) in our manuscript gives the exact variance \mathfrak{F} for a spatially-correlated input field. If we assume that relative strains have a Gaussian distribution $\bar{\mathcal{P}}(\mathbf{c}_{ij})$ of zero mean and variance \mathfrak{F} , the A in our patchy picture corresponds to the standard deviation $\sqrt{\mathfrak{F}}$, as illustrated in Fig. S3. Assuming that the correlator $f_\xi(x)$ is a Gaussian function (as in our Eq. (5)): if $\xi/\ell \ll 1$, A is essentially a constant; if $\xi/\ell \gg 1$, we have instead $A \sim \ell/\xi$. The latter case adds an additional dependence on ξ on the variance \mathfrak{F} , and thus on the typical strain felt by an interacting pair. That way we recover the crossover from $\mathfrak{F} \sim \ell/\xi$ to $\mathfrak{F} \sim (\ell/\xi)^3$ that we discuss in our manuscript.

In lower dimensions, these assumptions can be assumed to hold at least in the pre-yielding regime that we consider, on each elastic branch (albeit if we start to interfere with the spatial correlations of the response field after an AQRD step, as mentioned in our conclusion). In AQRD, after each minimization step, the random stress is given by our Eq. (4), as the scalar product of the forces acting on each particles \mathbf{F}_i and their respective vector \mathbf{c}_i . This can be rewritten as a scalar product of the forces between pairs and their respective relative strain \mathbf{c}_{ij} :

$$\sum_{\langle ij \rangle} \mathbf{F}_{ij} \cdot \mathbf{c}_{ij} = \frac{1}{2} \sum_{i,j=1}^N \mathbf{F}_{ij} \cdot (\mathbf{c}_i - \mathbf{c}_j) = \frac{1}{2} \sum_i \left(\sum_j \mathbf{F}_{ij} \right) \cdot \mathbf{c}_i + \frac{1}{2} \sum_j \left(\sum_i \mathbf{F}_{ji} \right) \cdot \mathbf{c}_j = \sum_i \mathbf{F}_i \cdot \mathbf{c}_i \propto -\tilde{\sigma}$$

where in the last equality we skipped the normalization with respect to the system size and the number of particles (see Eq.(4)). Let's assume that we start from a minimum after an AQRD step, described by a given set of relative position $\{\mathbf{r}_{ij}\}$. If we apply an infinitesimal strain increment $\Delta\tilde{\gamma}$ that keeps us in an elastic branch, we can Taylor-expand the forces using

$$\nabla v(|\mathbf{r}_{ij} + \Delta\tilde{\gamma} \mathbf{c}_{ij}|) = \nabla v(|\mathbf{r}_{ij}|) + \Delta\tilde{\gamma} \left[v''(r_{ij}) (\hat{\mathbf{r}}_{ij} \cdot \mathbf{c}_{ij}) \hat{\mathbf{r}}_{ij} + v'(r_{ij}) \frac{\mathbf{c}_{ij}}{r_{ij}} \right] + \mathcal{O}(\Delta\tilde{\gamma}^2).$$

The associated increase in random stress is then given by (we skip the finite-size normalization thereafter):

$$\begin{aligned} \Delta\tilde{\sigma} &\propto - \sum_{\langle ij \rangle} [\mathbf{F}_{ij}(\Delta\tilde{\gamma}) - \mathbf{F}_{ij}(\Delta\tilde{\gamma} = 0)] \cdot \mathbf{c}_{ij} \\ &= \Delta\tilde{\gamma} \sum_{\langle ij \rangle} \left[v''(r_{ij}) (\hat{\mathbf{r}}_{ij} \cdot \mathbf{c}_{ij}) \hat{\mathbf{r}}_{ij} + v'(r_{ij}) \frac{\mathbf{c}_{ij}}{r_{ij}} \right] \cdot \mathbf{c}_{ij} = \Delta\tilde{\gamma} \underbrace{\sum_{\langle ij \rangle} c_{ij}^2 \left[v''(r_{ij}) (\hat{\mathbf{r}}_{ij} \cdot \hat{\mathbf{c}}_{ij})^2 + \frac{v'(r_{ij})}{r_{ij}} \right]}_{\text{local elastic modulus}}. \end{aligned}$$

This local elastic modulus depends on the distribution of $\{\mathbf{r}_{ij}\}$, which is a highly nontrivial quantity to characterize analytically. In fact, the infinite-dimensional limit is the only case where we know it exactly, as a function of accumulated strain, which is one of the reasons why it is such a precious benchmark. Finally, if we assume that the typical local elastic modulus scales as the average c_{ij}^2 , by definition this is equal to the variance \mathfrak{F}/ℓ^2 . So we recover that the typical elastic modulus should scale as $\mu \sim \mathfrak{F}/\ell^2$, and thus inherits the ξ -dependence of \mathfrak{F} along the way. Note that we did not use any property specific to the Hertzian interaction potential, our argument holds for a generic soft potential.

References

1. P Morse, S Wijtmans, M van Deen, M van Hecke, ML Manning, Differences in plasticity between hard and soft spheres. *Phys. Rev. Res.* **2**, 023179 (2020).