Supplemental Information

Impact of Freezing Degrees of Freedom on Stability. Here, we show that freezing any subset of existing degrees of freedom in a system of particles can lead to an increase in the stability of the system. The two measuring tools we use are the distance to next instability, δP, and the density of states $\mathcal{D}(\omega)$ as discussed in Fig. 2. To measure δP, we first break the Hessian matrix into two terms, one that includes the rigidity matrix, $H_s = \mathbf{C}^T\mathbf{C}$, and one that includes the prestress forces, H_p. We then uniformly increase the prestress term using a dimensionless coefficient that includes the relative change in the pressure given by Eq. (3):

$$H = H_s + \frac{\delta P + P}{P}\ H_p \qquad [3]$$

Increasing δP in Eq. (3) is equivalent to increasing the contact forces uniformly. This pushes the system towards an instability without changing the geometrical configuration of the system. Since the prestress term in Eq. (3) is negative definite, increasing its components will push the eigenvalues of the Hessian matrix, H, to zero. To measure the distance to an instability, we increase δP in small steps and monitor the lowest non-zero eigenvalue of the Hessian. The δP for which this eigenvalue goes to zero is data that is presented in Figs. 2a,b, and 4a .

Fig. 4a shows the change in the pressure, δP, required to push a system to a nearby instability when the motion of all the particles is confined in direction x (red), in comparison to conventionally prepared packings (black) where the particles are free to move in all d available directions. The result is very similar to Fig. 2a, as freezing the x degrees of freedom increases the distance to a nearby instability by a few orders of magnitude.

Fig. 4b shows the ensemble averaged density of states in $3D$ systems when the x components of all particle positions are removed from the Hessian (red) compared to when all positional degrees of freedom are available (black). As can be seen from this plot, there is a clear shift to higher frequencies in the density of states which is similar to the shift in the lower tail of the density of states when radii were frozen at their equilibrium values in Fig. 2c. Both plots in Fig. 4 show that freezing an existing subset of degrees of freedom can make the system more stable.

Impact of Radius and Stiffness Degrees of Freedom on Rigidity. Here, we present a mathematical description for the difference between radius and stiffness degrees of freedom at the onset of rigidity. The main goal is to work out the type of degrees of freedom that can impact the Maxwell's count and shift the critical point. Since the Maxwell's count uses the rank-nullity theorem (50) on the rigidity matrix, we can start off by calculating this matrix which relates the changes in the constraints to changes in the available degrees of freedom (51, 52). For instance, in a packing with N particles and N_{con} overlapping contacts, changes in the overlaps are directly related to changes in the particle positions through Eq. (4):

$$\mathbf{C}\mathbf{u} = \mathbf{\Delta}h \qquad [4]$$

where $\mathbf{u}$ is an $Nd\times 1$ dimensional vector with elements that represent the changes in the positions of particles 1 through N, $\mathbf{C}$ is the rigidity matrix, and $\mathbf{\Delta}h$ is an $N_{con}\times 1$ dimensional vector where row n represents the change in the overlap between the nth pair of particles when they move. This means that the rigidity matrix, $\mathbf{C}$, is an $N_{con}\times Nd$ dimensional matrix.

Now, by adding the radius and stiffness degrees of freedom and a set of global constraints $\Phi_{\chi,m} = constant$, one can write a generalized version of Eq. (4) in the form of $\mathbf{C}\tilde{\mathbf{u}} = \mathbf{\Delta}h$ with $\tilde{\mathbf{u}}$ is the vector of displacements in the $N(d+2)-\mu$ dimensional space of all degrees of freedom where μ is the number of globally applied constraints ($\mu = 7$ in this paper). The $\mathbf{\Delta}h$ vector is the differential change in the overlaps, which for a pair ij is given by:

$$\begin{aligned} dh_{ij} = &\frac{\partial h_{ij}}{\partial x_i}dx_i + \frac{\partial h_{ij}}{\partial y_i}dy_i + \frac{\partial h_{ij}}{\partial z_i}dz_i + \frac{\partial h_{ij}}{\partial R_i}dR_i + \frac{\partial h_{ij}}{\partial K_i}dK_i \\ &+ \frac{\partial h_{ij}}{\partial x_j}dx_j + \frac{\partial h_{ij}}{\partial y_j}dy_j + \frac{\partial h_{ij}}{\partial z_j}dz_j + \frac{\partial h_{ij}}{\partial R_j}dR_j + \frac{\partial h_{ij}}{\partial K_j}dK_j \end{aligned} \qquad [5]$$

The setup for such a pair is shown in Fig. 5. Note that according to Eq. (1), the only variables that can change the overlaps and

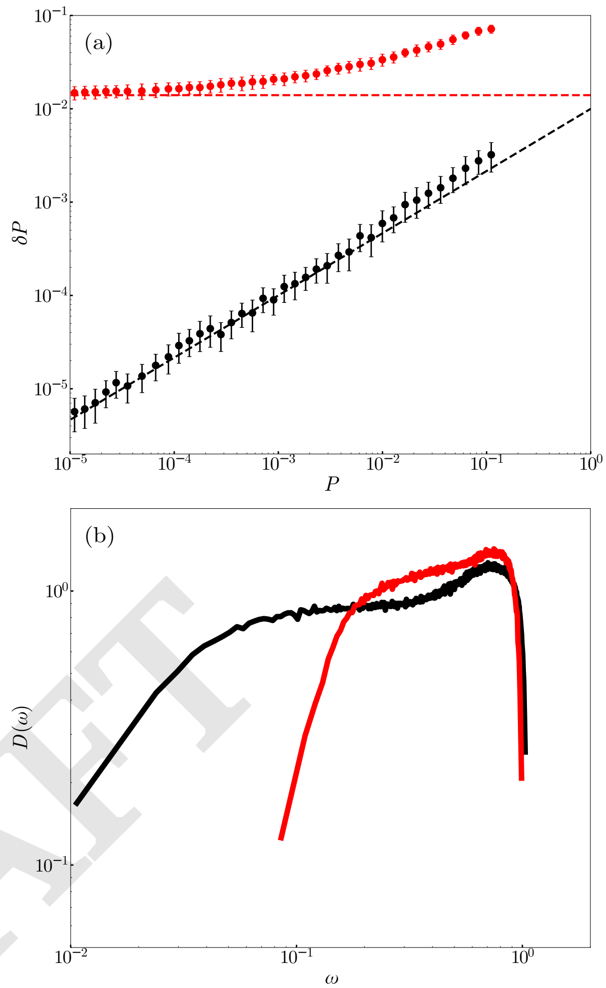

Fig. 4. a) The increase in pressure, δP, required to make a packing unstable before (black) and after freezing the x-components (red) of the positional degrees of freedom. The black dashed line show a power law $\delta P \propto P^{2/3}$. b) Density of states, $\mathcal{D}(\omega)$ versus ω, for a packing before (black) and after freezing the x-components (red) of the positional degrees of freedom. All the data in (a) and (b) are ensemble averaged over 20 mono disperse packings at pressure $p = 10^{-4}$.

therefore the number of rows in $\mathbf{\Delta}h$ are the positions $\{\mathbf{X_i}\}$ and radii $\{R_i\}$, since:

$$\begin{aligned} \frac{\partial h_{ij}}{\partial x_i} &= \frac{-1}{R_i+R_j}\frac{x_i - x_j}{|\boldsymbol{X}_i - \boldsymbol{X}_j|} \\ \frac{\partial h_{ij}}{\partial R_i} &= \frac{|\boldsymbol{X}_i - \boldsymbol{X}_j|}{(R_i+R_j)^2} = \beta_{ij} \\ \frac{\partial h_{ij}}{\partial K_i} &= 0 \end{aligned} \qquad [6]$$

By writing the vector $\tilde{\mathbf{u}}$ in the following form,

$$\mathbf{C} = \begin{array}{cc} & \begin{array}{ccccccc|ccccccc|ccccccc} 1 & \dots & i & \dots & j & \dots & N & 1 & \dots & i & \dots & j & \dots & N-\mu & 1 & \dots & i & \dots & j & \dots & N-\mu \end{array} \\ \begin{array}{c} \vdots \\ (i,j) \\ \vdots \end{array} & \left(\begin{array}{ccccccc|ccccccc|ccccccc} \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{n_{ij}} & \dots & \mathbf{n_{ji}} & \dots & \mathbf{0} & 0 & \dots & \beta_{ij} & \dots & \beta_{ji} & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \end{array} \right) \end{array} \qquad [8]$$

$$\tilde{\mathbf{u}} = \begin{bmatrix} dx_1 \\ dy_1 \\ dz_1 \\ \vdots \\ dz_N \\ dR_1 \\ dR_2 \\ dR_3 \\ \vdots \\ dR_{N-\mu} \\ dK_1 \\ dK_2 \\ dK_3 \\ \vdots \\ dK_{N-\mu} \end{bmatrix} \qquad [7]$$

one can obtain the generalized rigidity matrix with positions, radii, and particle stiffnesses as degrees of freedom:

where $\mathbf{n}_{ij} = \frac{-1}{R_i+R_j} \frac{\mathbf{X}_i - \mathbf{X}_j}{|\mathbf{X}_i - \mathbf{X}_j|} = -\mathbf{n}_{ji}$ and we assume that the number of global constraints on both the radius and stiffness degrees of freedom is equal to μ.

As can be seen from Eq. (8), adding the radii as degrees of freedom can change the rank of the rigidity matrix, rank($\mathbf{C}$), but adding the particle stiffnesses as degrees of freedom does not change rank($\mathbf{C}$) since it only adds $N-\mu$ extra columns of zeros to the matrix.

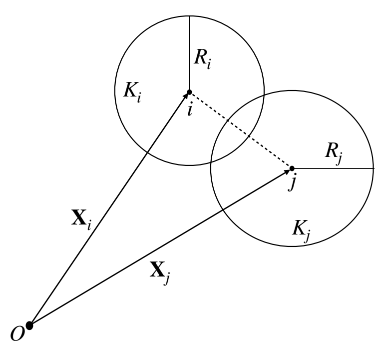

Fig. 5. Two particles i and j with positions $\mathbf{X}_i$ and $\mathbf{X}_j$, radii R_i and R_j, and stiffnesses K_i and K_j are shown. Changes in positions and radii of the particles can change their overlap.

A. Maxwell's count. Maxwell's count is the result of rank-nullity theorem on the rigidity matrix:

$$\begin{aligned} \text{rank}(\mathbf{C}) + \text{nullity}(\mathbf{C}) &= \text{number of columns in } \mathbf{C} \\ \text{rank}(\mathbf{C^T}) + \text{nullity}(\mathbf{C^T}) &= \text{number of columns in } \mathbf{C^T} \end{aligned} \qquad [9]$$

rank($\mathbf{C}$) = rank($\mathbf{C^T}$) is the number of independent rows/columns in $\mathbf{C}$ or $\mathbf{C^T}$. Nullity($\mathbf{C}$) is the dimension of the space of non-trivial solutions to $\mathbf{C}\tilde{\mathbf{u}} = \mathbf{0}$. This is the number of trivial rigid motions that do not change the overlaps and therefore is equal to the total number of zero modes or floppy modes, F, in the system. Nullity($\mathbf{C^T}$) on the other hand, is the number of non-trivial solutions to $\mathbf{C^T f} = \mathbf{0}$, which is equal to the number of states of self-stress, SSS. This is because $\mathbf{C^T f}$ gives the vector of forces on the particles and therefore solutions to $\mathbf{C^T f} = \mathbf{0}$ represent all the possible contact forces that keep the system at equilibrium. On the right hand side of the first equation in (9), we have number of columns in $\mathbf{C}$ which is equal to the number of degrees of freedom $Nd + 2(N-\mu)$. On the right hand side of the second equation in (9), we have number of columns in$\mathbf{C^T}$ which is equal to the number of contacts N_{con}. By subtracting the two rows in Eq. (9), one can write:

$$F = Nd + 2(N-\mu) - N_{con} + SSS \qquad [10]$$

which is the Maxwell's count for any system with position, radius, and stiffness degrees of freedom, N_{con} contacts, and μ global constraints. The number of contacts, N_{con}, can also be written as $N_{con} = ZN/2$ where Z is the average number of contacts per particle. The isostatic point is defined as the critical point in which the number of degrees of freedom is balanced by the number of constraints and there are no states of self-stress, meaning that $F = d$ and $SSS = \text{nullity}(\mathbf{C^T}) = 0$. In other words:

$$\begin{aligned} \frac{Z_c N}{2} &= \text{rank}(\mathbf{C^T}) \\ &= \text{rank}(\mathbf{C}) \\ &= \text{number of degrees of freedom} - \text{nullity}(\mathbf{C}) \end{aligned} \qquad [11]$$

B. Maxwell's count when positions are the only degrees of freedom. When the only degrees of freedom are positions, number of degrees of freedom $= Nd$ and nullity($\mathbf{C}$) $= d$. Therefore:

$$\frac{Z_c N}{2} = Nd - d \qquad [12]$$

which in the limit $N \to \infty$ gives $Z_c = 2d$.

C. Maxwell's count when radii are added as new degrees of freedom. Introducing radii as new degrees of freedom and μ global constraints adds $N-\mu$ new columns to the rigidity matrix and adds 1 to the nullity of the rigidity matrix:

$$\frac{Z_c N}{2} = N(d+1) - \mu - (d+1) \qquad [13]$$

which in the limit $N \to \infty$ gives $Z_c = 2(d+1)$. This means that radii can move the critical point to a higher value as long as μ is negligible compared to N.

D. Maxwell's count when stiffness of particles are added as new degrees of freedom. Introducing stiffnesses as new degrees of freedom and μ global constraints, adds $N-\mu$ new columns to the rigidity matrix. But since all these columns are zero, it does not change the rank of the rigidity matrix. Based on the rank-nullity theorem, this means that stiffness degrees of freedom add $N-\mu$ new zero modes to the system and increase the nullity($\mathbf{C}$) by $N-\mu$:

$$\begin{aligned} \frac{Z_c N}{2} &= N(d+1) - \mu - (d+N-\mu) \\ &= Nd - d \end{aligned} \qquad [14]$$

which in the limit $N \to \infty$ gives $Z_c = 2d$. This means that stiffness degrees of freedom do change the critical point simply because they do not appear in the definition of particle overlaps.