Supplementary Material

I. EXAMPLES OF DISTRIBUTIONS ν

The *Dirichlet* distribution for ν is specified by a choice of $k \in \mathbb{Z}_{\geq 1}$ and $\vec{\alpha} = (\alpha_{-k}, \dots, \alpha_k) \in \mathbb{R}_{>0}^{2k+1}$. It assigns probability density proportional to $\prod_{j=-k}^{k} \xi(j)^{\alpha_j}$ to each vector $\xi = (\xi(-k), \dots, \xi(k)) \in \mathbb{R}_{\geq 0}^{k}$ satisfying $\sum_{j=-k}^{k} \xi(j) = 1$. Then, we set $\xi(i) = 0$ for $i \notin [-k, k]$ so that ξ is defined on $\mathbb{Z}$. Note that a sample ξ under ν defines a probability distribution on $\mathbb{Z}$ due to the imposed normalization and non-negativity. When $\alpha_j \equiv 1$ for all $-k \leq j \leq k$, we call the resulting Dirichlet distribution *uniform*(k) since $(\xi(-k), \dots, \xi(k))$ is uniformly distributed over vectors in $\mathbb{R}_{\geq 0}^{k}$ that sum up to 1.

The *normalized i.i.d.* distribution for ν is specified by a choice of $k \in \mathbb{Z}_{\geq 1}$ and a choice of probability distribution on $\mathbb{R}_{\geq 0}$. Let $X_{-k}, \dots, X_k$ be chosen independently according to the specified probability distribution, and then define $\xi(i) = X_i (\sum_{j=-k}^{k} X_j)^{-1}$ for $i \in [-k, k]$ and $\xi(i) = 0$ otherwise. Such ξ are normalized to sum to 1 and are non-negative, and hence define a probability distribution on $\mathbb{Z}$. When the X_i are Gamma distributed with parameter α, the resulting measure on ξ matches the Dirichlet distribution with $\alpha_i \equiv \alpha$.

The *random delta* distribution for ν is specified by $k \in \mathbb{Z}_{\geq 1}$. Two numbers X_1, X_2 are drawn uniformly without replacement from $\{-k, \dots, k\}$. We set $\xi(X_1) = \xi(X_2) = 1/2$ and $\xi(i) = 0$ for all $i \neq X_1, X_2$.

II. CONVERGENCE TO THE STOCHASTIC HEAT EQUATION

We begin by summarizing the results in the forthcoming work [1]. These results also follow from [2]. We show that for $N \in \mathbb{Z}_{>0}$, $T \in N^{-1}\mathbb{Z}_{\geq 0}$ and $X \in (2DN)^{-1/2}(\mathbb{Z} - c_N T)$, the scaled moderate deviations of the tail probability for a single random walker converges as $N \to \infty$:

$$\frac{N^{1/4} C_{N,T,X}}{\sqrt{2D}} \mathbb{P}^{\boldsymbol{\xi}}(R^1(NT) \geq c_N T + \sqrt{2DN} X) \Rightarrow \tilde{Z}(T, X) \tag{S1}$$

with scaling parameters

$$C_{N,T,X} = \frac{\exp\left\{ \dfrac{c_N}{2DN^{1/4}} T + \dfrac{1}{\sqrt{2D}} N^{1/4} X \right\}}{\left(\displaystyle\sum_{i \in \mathbb{Z}} \mathbb{E}_\nu \left[\xi\left(i\right) \right] \exp\left\{ \dfrac{i}{2DN^{1/4}} \right\} \right)^{NT}}, \qquad c_N = N^{3/4} + \frac{\sum_{i \in Z} \mathbb{E}_\nu \left[\xi\left(i\right) \right] i^3}{2(2D)^2} N^{1/2}. \tag{S2}$$

The convergence is shown in [1] at the level of the first two moments (stronger process-level convergence is shown in [2]), and the limiting process $\tilde{Z}(T, X)$ is the solution to the multiplicative Stochastic Heat Equation (mSHE) with $\tilde{Z}(0, X) = \delta(X)$ initial data:

$$\partial_T \tilde{Z} = \frac{1}{2} \partial_X^2 \tilde{Z} + \sqrt{\frac{2\lambda_{\text{ext}}}{(2D)^{3/2}}} \tilde{Z} \eta. \tag{S3}$$

Here $\eta(T,X)$ is a space-time Gaussian white noise (i.e. $\mathbb{E}[\eta(T,X)]=0$ and $\mathbb{E}[\eta(T,X)\eta(T',X')] = \delta(X-X')\delta(T-T')$ where δ is the Dirac delta function). The noise strength is controlled by the Einstein diffusion coefficient D defined in (4) and λ_{ext}.

At the full level of generality of RWRE models we introduced in the *RWRE models* section, [1] (see also [2]) provides a somewhat involved formula for λ_{ext}:

$$\lambda_{\mathrm{ext}} := \frac{\mathrm{Var}_{\nu}\left(\mathbb{E}^{\boldsymbol{\xi}}[Y]\right)}{\sum_{l=0}^{\infty}\tilde{\mu}\left(l\right)\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]}. \tag{S4}$$

The numerator here is defined in terms of Y, a random jump distributed according to ξ. As in (10), this numerator is equal to $2D_{\mathrm{ext}}$. The denominator requires more explanation. Consider two random walks R^1 and R^2 distributed according to the measure $\mathbf{P}$ (i.e., after integrating over the law ν of the environment). These walks are not independent as they are coupled to the same (integrated out) environment—we call these the *two-point motions* as they have the law of two tracer particles when the environment is hidden. Define the gap between them to be

$$\Delta(t) := |R^1(t) - R^2(t)|.$$

There exists a unique (infinite mass) invariant measure for $V(t) := R^1(t) - R^2(t)$ and let $\mu(l)$ be the mass assigned to $l \in \mathbb{Z}$ with the normalization that $\mu(0) = 1$ (see e.g., [3] for background on invariant measures for Markov chains). The corresponding invariant measure for $\Delta(t)$ is therefore

$$\tilde{\mu}\left(l\right) := \begin{cases} 1 & \text{if } l=0 \\ 2\mu(l) & \text{if } l>0 \end{cases}. \tag{S5}$$

This invariant measure can be understood physically as follows. Start two particles near each other and let them diffuse in their common environment. After a relatively long time, measure the distance between them. Repeat this for many different environments, thus building up a histogram of the distances between these two-point motions. The typical distance will be large, but if we cutoff to consider relatively short distances, and normalize the histogram to put weight 1 at distance 0, then it will converge to $\tilde{\mu}\left(l\right)$. A slightly different way that this same measure should arise is from surveying the inter-particle distances over all particles in a many-particle diffusion. Normalizing the histogram of these distances to be 1 at distance 0, will yield a histogram that converges to $\tilde{\mu}\left(l\right)$ at distance l. This should hold for a single environment since the inter-particle distances (on the short-scale that we are considering) will generally feel different parts of the environment, and hence experience some averaging.

The denominator in (S4) is independent of t and it has the interpretation as the expected change in the two-point motion gap when started under its invariant measure. For large enough l (since we have assumed a finite range for the jump distribution) the expected change of the gap will be zero and hence the sum will be a finite one.

We will show that λ_{ext} simplifies considerably for a wide class of distributions ν in Section VI, and (before that) in the next section we will explain (in the spirit of [4, 5]) how we go from this mSHE convergence result to our extreme diffusion theoretical predictions.

However, let us first briefly summarize the approach used in [1] to derive this mSHE convergence result. First of all, (S1) is only demonstrated therein at the level of convergence of first and second moments. Convergence of higher moments follows similarly as noted below. Since the moments of the mSHE do not characterize its distribution, more work is needed, as in [2], to show convergence in distribution, or convergence of the space-time process.

The second moment of the LHS in (S1) can be expressed via a discrete Feynman-Kac formula in terms of the expectation of the exponential of the self-intersection time for the two-point motion (R^1, R^2) under a tilting of the measure $\mathbf{P}$ (high integer moments involve higher order self-intersection times). For the mSHE, all integer moments can be expressed in terms of the expectation of the exponential of the pair local times at zero for independent Brownian motions (this is the replica method, see [6, 7]). The k-point motion converges diffusively to k independent Brownian motions, yet the discrete self-intersection time does not converge to the local time at zero for the Brownian motions.

This failure of self-intersection time to local time convergence may seem surprising, but a very simple example should convince the reader that such convergence is more subtle. For instance, imagine two independent SSRWs, one started at 0 and one started at 1. They converge jointly to two Brownian motions, yet their intersection time is always 0 (they are on different sublattices) and hence does not converge to the local time at 0 for the Brownian motions.

A discrete version of the Tanaka formula—which, for a Brownian motion B, shows that $|B(t)| = \int_0^t \mathrm{sgn}(B(s))dB(s) + L(t)$ where $\mathrm{sgn}(x)$ is the derivative of the absolute value function $|x|$ (set to be 0 at 0) and where $L(t)$ is the local time at zero up to time t—is used in [1] to identify the discrete quantity that converges to the Brownian local time $L(t)$. That discrete local time involves reweighting the expected change in distance between the two-point motion according to its invariant measure. This allows us to identify the limit of the self-intersection time and leads to the denominator in (S4).

III. APPROXIMATING THE TAIL PROBABILITY

After taking logarithms in (S1), we see that as $N \to \infty$

$$\ln(\mathbb{P}^{\boldsymbol{\xi}}(R^1(NT) \geq c_N T + \sqrt{2DN}X)) \\ \approx -\frac{c_N T}{2DN^{1/4}} - \frac{N^{1/4}X}{\sqrt{2D}} + NT\ln\left(\sum_{i\in\mathbb{Z}} \mathbb{E}_\nu\left[\xi\left(i\right)\right]\exp\left\{\frac{i}{2DN^{1/4}}\right\}\right) - \ln\left(\frac{N^{1/4}}{\sqrt{2D}}\right) + \tilde{h}(T,X) \tag{S6}$$

where $\tilde{h}(T,X) = \ln(\tilde{Z}(T,X))$ solves the KPZ equation with narrow wedge initial data and with λ_{ext}-dependent noise strength (we now also include tildes on the X and T variables to simplify the below change of variables),

$$\partial_{\tilde{T}}\tilde{h} = \frac{1}{2}\partial_{\tilde{X}}^2\tilde{h} + \frac{1}{2}(\partial_{\tilde{X}}\tilde{h})^2 + \sqrt{\frac{2\lambda_{\mathrm{ext}}}{(2D)^{3/2}}}\eta.$$

Defining $h(T,X) = \tilde{h}(\tilde{T},\tilde{X})$ with $T = \frac{4\lambda_{\mathrm{ext}}^2}{(2D)^3}\tilde{T}$ and $X = \frac{2\lambda_{\mathrm{ext}}}{(2D)^{3/2}}\tilde{X}$, h solves the standard coefficient KPZ equation

$$\partial_T h = \frac{1}{2}\partial_X^2 h + \frac{1}{2}(\partial_X h)^2 + \eta.$$

We also have that as $N \to \infty$,

$$\left(\sum_{i\in\mathbb{Z}} \mathbb{E}_\nu\left[\xi\left(i\right)\right]\exp\left\{\frac{i}{2DN^{1/4}}\right\}\right) \approx \frac{N^{-1/2}}{4D} + \frac{\sum_{i\in\mathbb{Z}}\mathbb{E}_\nu\left[\xi\left(i\right)\right]i^3}{6(2D)^3}N^{-3/4} + \mathcal{O}\left(N^{-1}\right).$$

Substituting this into (S6) and using our transformation to the KPZ equation, we find

$$\ln\left(\mathbb{P}^{\boldsymbol{\xi}}\left(R^1(NT) \geq N^{3/4}T + \frac{m_3 N^{1/2}T}{2(2D)^2} + \sqrt{2DN}X\right)\right) \approx -\frac{N^{1/2}T}{4D} - \frac{N^{1/4}X}{\sqrt{2D}} - \frac{N^{1/4}Tm_3}{3(2D)^3} + \ln\left(\frac{N^{1/4}}{\sqrt{2D}}\right) + h\left(\frac{4\lambda_{\text{ext}}^2}{(2D)^3}T, \frac{2\lambda_{\text{ext}}}{(2D)^{3/2}}X\right) + \mathcal{O}(T)$$

where $m_3 = \sum_{i\in\mathbb{Z}} \mathbb{E}_\nu [\xi(i)]\, i^3$. We now introduce the time $t := NT$, velocity $v := T^{1/4}$, and rescaled position $y = \frac{X}{v^2}$. Making these substitutions, we find

$$\ln\left(\mathbb{P}^{\boldsymbol{\xi}}\left(R^1(t) \geq vt^{3/4} + \frac{m_3}{2(2D)^2}v^2 t^{1/2} + \sqrt{2Dt}y\right)\right) \approx -\frac{v^2}{4D}t^{1/2} - \frac{vy}{\sqrt{2D}}t^{1/4} - \frac{v^3 m_3}{3(2D)^3}t^{1/4} + \ln\left(\frac{t^{1/4}}{\sqrt{2D}v}\right) + h\left(\frac{4\lambda_{\text{ext}}^2}{(2D)^3}v^4, \frac{2\lambda_{\text{ext}}}{(2D)^{3/2}}yv^2\right) + \mathcal{O}(v^4). \tag{S7}$$

IV. EXTREME FIRST PASSAGE TIME THEORETICAL PREDICTIONS

Here we derive asymptotic predictions for the means and variances of Env_L^N, Sam_L^N and Min_L^N. Much of this analysis follows from [5], where they derived predictions for the nearest neighbor case with a uniform distribution.

We start by substituting $y = 0$ and $L = vt^{3/4}$ into (S7) such that

$$\ln\left(\mathbb{P}^{\boldsymbol{\xi}}\left(R^1(t) \geq L + \frac{m_3 L^2}{2(2D)^2 t}\right)\right) \approx -\frac{L^2}{4Dt} - \frac{m_3 L^3}{3(2D)^3 t^2} + \ln\left(\frac{t}{\sqrt{2D}L}\right) + h\left(\frac{4\lambda_{\text{ext}}^2 L^4}{(2D)^3 t^3}, 0\right) + \mathcal{O}\left(\frac{L^4}{t^3}\right). \tag{S8}$$

We can now drop all subdominant terms as they will not contribute to the asymptotic predictions (though they could offer some higher-order corrections that we do not probe here). This yields a rougher approximation where we do not track the order of the error

$$\ln\left(\mathbb{P}^{\boldsymbol{\xi}}\left(R^1(t) \geq L\right)\right) \approx -\frac{L^2}{4Dt} + h\left(\frac{4\lambda_{\text{ext}}^2 L^4}{(2D)^3 t^3}, 0\right). \tag{S9}$$

We now utilize the non-backtracking approximation $\mathbb{P}^{\boldsymbol{\xi}}(R^1(t) \geq L) \approx \mathbb{P}^{\boldsymbol{\xi}}(\tau_L \leq t)$ (as discussed in [5]) as L gets large to yield

$$\ln\left(\mathbb{P}^{\boldsymbol{\xi}}\left(\tau_L \leq t\right)\right) \approx -\frac{L^2}{4Dt} + h\left(\frac{4\lambda_{\text{ext}}^2 L^4}{(2D)^3 t^3}, 0\right). \tag{S10}$$

We now substitute Env_L^N into this equation, recalling that Env_L^N is approximately the time t such that $\mathbb{P}^{\boldsymbol{\xi}}(\tau_L \leq t) = 1/N$ (in fact, the minimum time t satisfying $\mathbb{P}^{\boldsymbol{\xi}}(\tau_L \leq t) \geq 1/N$ though this difference is negligible):

$$-\ln(N) \approx -\frac{L^2}{4D\text{Env}_L^N} + h\left(\frac{4\lambda_{\text{ext}}^2 L^4}{(2D)^3\left(\text{Env}_L^N\right)^3}, 0\right). \tag{S11}$$

We can solve this equation perturbatively for Env_L^N. The term $-\frac{L^2}{4D\mathrm{Env}_L^N}$ in (S11) is dominant for large L, which yields the first-order estimate

$$\mathrm{Env}_L^N \approx T_L^N := \frac{L^2}{4D\ln(N)}. \tag{S12}$$

We now consider a small perturbation about T_L^N such that $\mathrm{Env}_L^N = T_L^N + \delta$ where $\delta \ll T_L^N$ contains the randomness of Env_L^N. Substituting this into (S11), we find

$$-\ln(N) \approx -\frac{L^2}{4D(T_L^N+\delta)} + h\left(\frac{4\lambda_{\mathrm{ext}}^2 L^4}{(2D)^3\left(T_L^N+\delta\right)^3}, 0\right). \tag{S13}$$

Since $\delta \ll T_L^N$, we approximate $-\frac{L^2}{4D(T_L^N+\delta)} \approx -\frac{L^2}{4DT_L^N} + \frac{L^2}{4D(T_L^N)^2}\delta$ and $\frac{4\lambda_{\mathrm{ext}}^2 L^4}{(2D)^3\left(T_L^N+\delta\right)^3} \approx \frac{4\lambda_{\mathrm{ext}}^2 L^4}{(2D)^3\left(T_L^N\right)^3}$. As such we can solve for δ, yielding

$$\delta \approx -\frac{L^2}{4D\ln(N)^2} \cdot h\left(\frac{4\lambda_{\mathrm{ext}}^2 L^4}{(2D)^3(T_L^N)^3}, 0\right) = -\frac{L^2}{4D\ln(N)^2} \cdot h\left(\frac{32\lambda_{\mathrm{ext}}^2(\ln(N))^3}{L^2}, 0\right).$$

Therefore, the mean and variance of Env_L^N is given by

$$\mathbf{E}\left[\mathrm{Env}_L^N\right] \approx \frac{L^2}{4D\ln(N)} \tag{S14}$$

$$\mathbf{Var}\left(\mathrm{Env}_L^N\right) \approx \frac{L^4}{(4D)^2\ln(N)^4}\mathbf{Var}\left(h\left(\frac{32\lambda_{\mathrm{ext}}^2(\ln(N))^3}{L^2}, 0\right)\right). \tag{S15}$$

We now consider the limit where $L \gg (\ln(N))^{3/2}$. In this limit, we simplify (S15) using the small-time KPZ Gaussian approximation

$$h(s,0) \approx -\frac{s}{24} - \ln\left(\sqrt{2\pi s}\right) + \left(\frac{\pi s}{4}\right)^{1/4} G_s \tag{S16}$$

where G_s converges as $s \to 0$ to a standard Gaussian, see for instance [8, 9]. Thus, for $L \gg \lambda_{\mathrm{ext}}(\ln(N))^{3/2}$ we find

$$\mathbf{Var}\left(\mathrm{Env}_L^N\right) \approx \lambda_{\mathrm{ext}}\frac{\sqrt{2\pi}L^3}{8D^2\ln(N)^{5/2}}. \tag{S17}$$

We now derive the distribution, and subsequently the mean and variance, for the randomness due to sampling random walks, Sam_L^N. We begin by using the approximation, $(1+x)^N \approx e^{xN}$ for $x \ll 1$ and $N \to \infty$. Therefore, for $N \to \infty$ and small $\mathbb{P}^{\boldsymbol{\xi}}(\tau_L \le t)$, we approximate (15) as

$$\ln\left(\mathbb{P}^{\boldsymbol{\xi}}(\mathrm{Sam}_L^N > t)\right) \approx -N\mathbb{P}^{\boldsymbol{\xi}}(\tau_L \le t + \mathrm{Env}_L^N). \tag{S18}$$

Substituting (S10) yields

$$\ln\left(\mathbb{P}^{\boldsymbol{\xi}}(\mathrm{Sam}_L^N > t)\right) \approx -N\exp\left\{-\frac{L^2}{4D(t+\mathrm{Env}_L^N)}\right\} \tag{S19}$$

where we have dropped all but the leading order term. We now assume $\mathrm{Env}_L^N \gg t$, which is justified because, as we will show, $\mathbf{E}\left[\mathrm{Env}_L^N\right] \gg \mathbf{E}\left[\mathrm{Sam}_L^N\right]$. When $\mathrm{Env}_L^N \gg t$, we approximate $(t+\mathrm{Env}_L^N)^{-1} \approx \frac{1}{\mathrm{Env}_L^N}\left(1-\frac{t}{\mathrm{Env}_L^N}\right)$ such that

$$\ln\left(\mathbb{P}^{\boldsymbol{\xi}}(\mathrm{Sam}_L^N > t)\right) \approx -N\exp\left\{-\frac{L^2}{4D\mathrm{Env}_L^N} - \frac{tL^2}{4D\left(\mathrm{Env}_L^N\right)^2}\right\}. \tag{S20}$$

Now replacing Env_L^N with its leading order approximation, T_L^N, from (S12), we find

$$\ln\left(\mathbb{P}^{\boldsymbol{\xi}}(\mathrm{Sam}_L^N \leq t)\right) \approx -\exp\left\{-\frac{4D\ln(N)^2 t}{L^2}\right\}.$$

By replacing Env_L^N with T_L^N we have assumed that Sam_L^N and Env_L^N are independent since Sam_L^N no longer depends on the randomness of Env_L^N. Though likely theoretically justifiable, this approximation is a fortiori justified by our numerics as shown in Figure 2. From this we find that $-\mathrm{Sam}_L^N$ is Gumbel distributed with mean and variance

$$\mathbf{E}[\mathrm{Sam}_L^N] \approx -\frac{\gamma L^2}{4D\ln(N)^2} \tag{S21}$$

$$\mathbf{Var}(\mathrm{Sam}_L^N) \approx \frac{\pi^2 L^4}{96D^2\ln(N)^4} \tag{S22}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. Notice that as L and N tend to infinity with $L \gg \ln(N)^{3/2}$, $\mathbf{E}\left[\mathrm{Sam}_L^N\right] \ll \mathbf{E}\left[\mathrm{Env}_L^N\right]$ which justifies the approximation used in (S20).

We solve for the mean and variance of Min_L^N by rearranging our definition of Sam_L^N such that $\mathrm{Min}_L^N = \mathrm{Sam}_L^N + \mathrm{Env}_L^N$. Since $\mathbf{E}\left[\mathrm{Sam}_L^N\right] \ll \mathbf{E}\left[\mathrm{Env}_L^N\right]$,

$$\mathbf{E}\left[\mathrm{Min}_L^N\right] \approx \frac{L^2}{4D\ln(N)}. \tag{S23}$$

Since Env_L^N and Sam_L^N are independent, $\mathbf{Var}\left(\mathrm{Min}_L^N\right) = \mathbf{Var}\left(\mathrm{Sam}_L^N\right) + \mathbf{Var}\left(\mathrm{Env}_L^N\right)$ such that

$$\mathbf{Var}\left(\mathrm{Min}_L^N\right) \approx \frac{\pi^2 L^4}{96D^2\ln(N)^4} + \lambda_{\mathrm{ext}}\frac{\sqrt{2\pi}L^3}{8D^2\ln(N)^{5/2}} \tag{S24}$$

in the limit $L \gg \ln(N)^{3/2}$.

V. EXTREME LOCATION THEORETICAL PREDICTIONS

Here we derive asymptotic predictions for the means and variances of Env_t^N, Sam_t^N and Max_t^N. As above, much of this analysis follows from [4], where they derived predictions for the nearest neighbor case with a uniform distribution. The argument proceeds similarly to the first passage time analysis. Starting at (S9), we substitute Env_t^N for the position L such that $\mathbb{P}^{\boldsymbol{\xi}}(R^1(t) \geq L) = 1/N$, thus yielding

$$-\ln(N) \approx -\frac{\left(\mathrm{Env}_t^N\right)^2}{4Dt} + h\left(\frac{4\lambda_{\mathrm{ext}}^2\left(\mathrm{Env}_t^N\right)^4}{(2D)^3 t^3}, 0\right). \tag{S25}$$

The first term on the right-hand side dominates and yields the first-order estimate

$$\mathrm{Env}_t^N \approx X_t^N := \sqrt{4Dt\ln(N)}. \tag{S26}$$

We consider a small perturbation, δ, about X_t^N such that $\mathrm{Env}_t^N = X_t^N + \delta$ where $\delta \ll X_t^N$. Substituting this into (S25) and only taking the highest order terms yields

$$\delta \approx \sqrt{\frac{Dt}{\ln(N)}}\, h\left(\frac{8\lambda_{\mathrm{ext}}^2\ln(N)^2}{Dt}, 0\right). \tag{S27}$$

Therefore, the mean and variance of Env_t^N satisfy

$$\mathbf{E}\left[\mathrm{Env}_t^N\right] \approx \sqrt{4Dt\ln(N)} \tag{S28}$$

$$\mathbf{Var}\left(\mathrm{Env}_t^N\right) \approx \frac{Dt}{\ln(N)}\mathbf{Var}\left(h\left(\frac{8\lambda_{\mathrm{ext}}^2\ln(N)^2}{Dt}, 0\right)\right). \tag{S29}$$

When $t \gg \frac{\lambda_{\mathrm{ext}}^2}{D}\ln(N)^2$, we use the small argument expansion of the KPZ equation in (S16) to approximate

$$\mathbf{Var}\left(\mathrm{Env}_t^N\right) \approx \lambda_{\mathrm{ext}}\sqrt{2\pi Dt}. \tag{S30}$$

We now derive the distribution for the randomness due to sampling random walks, Sam_t^N. For $N \to \infty$ and small $\mathbb{P}^{\boldsymbol{\xi}}(R(t) \leq x + \mathrm{Env}_t^N)$, we approximate (16) as

$$\ln(\mathbb{P}^{\boldsymbol{\xi}}(\mathrm{Sam}_t^N \leq x)) \approx -N\mathbb{P}^{\boldsymbol{\xi}}(R(t) \leq x + \mathrm{Env}_t^N). \tag{S31}$$

Substituting (S9) yields

$$\ln(\mathbb{P}^{\boldsymbol{\xi}}(\mathrm{Sam}_t^N \leq x)) \approx -N\exp\left\{-\frac{(x+\mathrm{Env}_t^N)^2}{4Dt}\right\} \tag{S32}$$

where we have only kept the leading order term of $\mathbb{P}^{\boldsymbol{\xi}}(R(t) \leq x)$. We assume $\mathrm{Env}_t^N \gg x$ which is justified because, as we will show, $\mathbf{E}\left[\mathrm{Env}_t^N\right] \gg \mathbf{E}\left[\mathrm{Sam}_t^N\right]$. We use this to approximate $(x + \mathrm{Env}_t^N)^2 \approx (\mathrm{Env}_t^N)^2 + 2\mathrm{Env}_t^N x$ such that

$$\ln(\mathbb{P}^{\boldsymbol{\xi}}(\mathrm{Sam}_t^N \leq x)) \approx -N\exp\left\{-\frac{(\mathrm{Env}_t^N)^2}{4Dt} - \frac{\mathrm{Env}_t^N x}{2Dt}\right\}. \tag{S33}$$

Replacing Env_t^N with its first-order approximation, X_t^N, in (S26), we find

$$\ln(\mathbb{P}^{\boldsymbol{\xi}}(\mathrm{Sam}_t^N \leq x)) \approx -\exp\left\{-\sqrt{\frac{\ln(N)}{Dt}}x\right\}. \tag{S34}$$

Therefore, Sam_t^N is Gumbel distributed with mean and variance

$$\mathbf{E}[\mathrm{Sam}_L^N] \approx \gamma\sqrt{\frac{Dt}{\ln(N)}} \tag{S35}$$

$$\mathbf{Var}(\mathrm{Sam}_L^N) \approx \frac{\pi^2 Dt}{6\ln(N)}. \tag{S36}$$

Notice that as $t \to \infty$, $\mathbf{E}\left[\mathrm{Env}_t^N\right] \gg \mathbf{E}\left[\mathrm{Sam}_t^N\right]$ which justifies our approximation in (S33).

We solve for the mean and variance of Max_t^N by rearranging our definition of Sam_t^N such that $\mathrm{Max}_t^N = \mathrm{Sam}_t^N + \mathrm{Env}_t^N$. Since $\mathbf{E}\left[\mathrm{Env}_t^N\right] \gg \mathbf{E}\left[\mathrm{Sam}_t^N\right]$,

$$\mathbf{E}\left[\mathrm{Max}_t^N\right] \approx \sqrt{4Dt\ln(N)}. \tag{S37}$$

Since Env_t^N and Sam_t^N are assumed to be asymptotically independent, $\mathbf{Var}\left(\mathrm{Max}_t^N\right) = \mathbf{Var}\left(\mathrm{Env}_t^N\right) + \mathbf{Var}\left(\mathrm{Sam}_t^N\right)$ so

$$\mathbf{Var}\left(\mathrm{Max}_t^N\right) \approx \frac{\pi^2 Dt}{6\ln(N)} + \lambda_{\mathrm{ext}}\sqrt{2\pi Dt} \tag{S38}$$

when $t \gg \lambda_{\mathrm{ext}}^2 \ln(N)^2$.

VI. CALCULATING THE COEFFICIENT FOR SEVERAL DISTRIBUTIONS

We demonstrate that the coefficient λ_{ext} in the mSHE/KPZ equation limit (S1) simplifies to the expression in (9) for a class of distributions ν including those introduced earlier (i.e., the Dirichlet, normalized i.i.d., and random delta distributions), as well as all nearest neighbor distributions.

A. General Model

We study the following class of distributions such that:

There exists some $c \in (0,1)$ such that for all $i \neq j$ $\qquad \mathbb{E}_\nu\left[\xi(i)\xi(j)\right] = c\mathbb{E}_\nu\left[\xi(i)\right]\mathbb{E}_\nu\left[\xi(j)\right].$ (S39)

We will also initially assume that the difference walk $V(t) = R^1(t) - R^2(t)$ (where R^1 and R^2 are distributed according to $\mathbf{P}$) is irreducible (i.e., $V(t)$ can reach any location on $\mathbb{Z}$ when started from 0), although we will later also consider the nearest neighbor model in Section VI E, which does not satisfy this condition as the difference walk $V(t)$ in that case is restricted to the even integer sublattice. A sufficient condition for $V(t)$ to be irreducible is that

$$\mathbb{E}_\nu\left[\xi\left(i\right)\right] > 0 \text{ for all } i \in \{-1, 0, 1\}. \tag{S40}$$

We compute λ_{ext} by simplifying the numerator and denominator of (S4) in terms of c and then matching this to (9). The numerator of (S4) simplifies to

$$\mathrm{Var}_\nu\left(\mathbb{E}^{\boldsymbol{\xi}}[Y]\right) = \sum_{i\in\mathbb{Z}}(1-c)\mathbb{E}_\nu\left[\xi(i)\right]i^2. \tag{S41}$$

Simplifying the denominator of (S4) is much more involved. We do this in three steps. First, we compute the invariant measure, $\tilde{\mu}\left(l\right)$ and show that

$$\tilde{\mu}\left(l\right) = \begin{cases} 1 & \text{if } i = 0 \\ 2c & \text{if } i \neq 0 \end{cases}. \tag{S42}$$

Then we simplify the expression $\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]$ and show that (without using the assumption (S39))

$$\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]=\begin{cases}\displaystyle\sum_{i,j\in\mathbb{Z}}|i-j|\mathbb{E}_{\nu}\left[\xi(i)\xi(j)\right] & l=0\\ \displaystyle\sum_{|i-j|>l}(|i-j|-l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right] & l>0\end{cases}. \tag{S43}$$

Lastly, we combine these two results to compute the sum over l and show that

$$\sum_{l=0}^{\infty}\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]\tilde{\mu}\left(l\right)=2c\sum_{i\in\mathbb{Z}}i^{2}\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]. \tag{S44}$$

The invariant measure: We now compute the invariant measure for the walk $V(t)$. The transition probabilities of V are given by

$$p(i,j):=\mathbf{P}(V(t+1)=j\mid V(t)=i)=\begin{cases}\sum_{k\in\mathbb{Z}}\mathbb{E}_{\nu}\left[\xi(k)\xi(k-j)\right] & \text{if } i=0\\ \sum_{k\in\mathbb{Z}}\mathbb{E}_{\nu}\left[\xi\left(k\right)\right]\mathbb{E}_{\nu}\left[\xi\left(k-j+i\right)\right] & \text{if } i\neq 0.\end{cases}. \tag{S45}$$

Notice that if $i,j\neq 0$, then by a change of variables, we have

$$\begin{aligned}p(i,j)&=\sum_{k\in\mathbb{Z}}\mathbb{E}_{\nu}\left[\xi\left(k-j+i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(k\right)\right]\\&=\sum_{\tilde{k}\in\mathbb{Z}}\mathbb{E}_{\nu}\left[\xi\left(\tilde{k}\right)\right]\mathbb{E}_{\nu}\left[\xi\left(\tilde{k}-i+j\right)\right]\\&=p(j,i).\end{aligned} \tag{S46}$$

It also follows from (S39) that

$$p(0,j)=cp(j,0). \tag{S47}$$

We claim that the unique invariant measure of $V(t)$ (up to multiplication by a constant coefficient) is given by $\mu(l)=c$ for all $l\neq 0$ and $\mu(0)=1$. We check this by showing $\mu(\cdot)$ satisfies the detailed balance equation,

$$\mu(i)p(i,j)=\mu(j)p(j,i) \tag{S48}$$

for all $i,j\in\mathbb{Z}$. This also shows that the walk $V(t)$ is reversible with respect to $\mu(\cdot)$.

The detailed balance condition is clearly true for the case $i=j=0$, and for $i,j\neq 0$ it follows from (S46). We now consider the case when $i=0$ and $j\neq 0$. Substituting $\mu(j)=c$ and $\mu(0)=1$ into (S48), we obtain

$$p(0,j)=cp(j,0),$$

which is true by (S47). Therefore, our claim that $\mu(l)=c$ for $l\neq 0$ and $\mu(0)=1$ is justified.

It follows from (S5) that

$$\tilde{\mu}\left(l\right)=\begin{cases}1 & \text{if } i=0\\ 2c & \text{if } i\neq 0\end{cases} \tag{S49}$$

as desired.

Simplification of the expectation value: We now simplify $\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]$ to show it is given by (S43). For $l=0$, the two walkers R^1 and R^2 are both at the same site; therefore, they use the same jump rate such that

$$\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=0\right]=\sum_{i,j\in\mathbb{Z}}|i-j|\mathbb{E}_{\nu}\left[\xi(i)\xi(j)\right],$$

which agrees with (S43).

For $l>0$, the two walkers R^1 and R^2 are at different sites; therefore, they use independent jump rates such that

$$\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]=\sum_{i,j\in\mathbb{Z}}(|l+i-j|-l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]. \tag{S50}$$

We now break this sum up into two parts: when $|i-j|\leq l$ and $|i-j|>l$. For $|i-j|\leq l$, we simplify

$$|l+i-j|-l=\begin{cases}i-j & \text{if } i\geq j\\ j-i & \text{if } i<j,\end{cases}.$$

Therefore,

$$\begin{aligned}\sum_{|i-j|\leq l}(|l+i-j|-l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]&=\sum_{i<j}(j-i)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]+\sum_{i\geq j}(i-j)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]\\&=0\end{aligned}$$

after swapping the indices of the second sum.

For $|i-j|>l$, we find

$$|l+i-j|-l=\begin{cases}i-j & \text{if } i\geq j\\ j-i-2l & \text{if } i<j\end{cases}.$$

Substituting this into (S50), we are left with

$$\begin{aligned}\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]&=\sum_{|i-j|>l}(|l+i-j|-l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]\\&=\sum_{i-j>l}(|i-j|)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]+\sum_{j-i>l}(|i-j|-2l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]\\&=\sum_{|i-j|>l}(|i-j|-l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right].\end{aligned}$$

Thus, $\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]$ is given by (S43).

Computing λ_{ext}: We now compute λ_{ext} using our formula for $\tilde{\mu}\left(l\right)$ and

$\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]$. Substituting (S49) and (S43), we find

$$\sum_{l=0}^{\infty}\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]\tilde{\mu}\left(l\right)=c\sum_{i,j\in\mathbb{Z}}|i-j|\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]+2c\sum_{l=1}^{\infty}\sum_{|i-j|>l}(|i-j|-l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right] \tag{S51}$$

$$=c\sum_{l=0}^{\infty}\sum_{|i-j|>l}(2-\mathbb{1}_{l=0})(|i-j|-l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right] \tag{S52}$$

after combining the two sums. Now we break up the second sum over i and j such that

$$\begin{aligned}&\sum_{l=0}^{\infty}\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]\tilde{\mu}\left(l\right)\\&\quad=c\sum_{l=0}^{\infty}\sum_{j\in\mathbb{Z}}\left[\sum_{i=j+l+1}^{\infty}(2-\mathbb{1}_{l=0})(i-j-l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]+\sum_{i=-\infty}^{j-l-1}(2-\mathbb{1}_{l=0})(j-i-l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]\right]\\&\quad=c\sum_{j\in\mathbb{Z}}\left[\sum_{i=j+1}^{\infty}\sum_{l=0}^{i-j-1}(2-\mathbb{1}_{l=0})(i-j-l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]+\sum_{i=-\infty}^{j-1}\sum_{l=0}^{j-i-1}(2-\mathbb{1}_{l=0})(j-i-l)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]\right],\end{aligned}$$

where in the second step, we switch the order of the sums (which is justified by our assumption that ξ is finite range). Notice that since $\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]$ does not depend on l we can now evaluate the sums over l. This yields

$$\begin{aligned}\sum_{l=0}^{\infty}\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]\tilde{\mu}\left(l\right)&=c\sum_{j\in\mathbb{Z}}\left[\sum_{i=j+1}^{\infty}(i-j)^2\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]+\sum_{i=-\infty}^{j-1}(i-j)^2\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]\right]\\&=c\sum_{i,j\in\mathbb{Z}}(i-j)^2\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]\\&=c\sum_{i,j\in\mathbb{Z}}(i^2+j^2)\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\mathbb{E}_{\nu}\left[\xi\left(j\right)\right]\\&=2c\sum_{i\in\mathbb{Z}}i^2\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]\end{aligned}$$

after using $\sum_{i\in\mathbb{Z}}\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]i=0$. Thus, we find

$$\lambda_{\mathrm{ext}}=\frac{\mathrm{Var}_{\nu}\left(\mathbb{E}^{\boldsymbol{\xi}}[Y]\right)}{\sum_{l=0}^{\infty}\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]\tilde{\mu}\left(l\right)}=\frac{1-c}{2c}. \tag{S53}$$

We now match this to (9). We recall that $D=\frac{1}{2}\sum_{i\in\mathbb{Z}}i^2\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]$ and that $D_{\mathrm{ext}}=\frac{1}{2}\mathrm{Var}_{\nu}\left(\mathbb{E}^{\boldsymbol{\xi}}[Y]\right)$ where $\mathrm{Var}_{\nu}\left(\mathbb{E}^{\boldsymbol{\xi}}[Y]\right)$ is given in (S41). Therefore,

$$\frac{D_{\mathrm{ext}}}{2(D-D_{\mathrm{ext}})}=\frac{1-c}{2c}=\lambda_{\mathrm{ext}}. \tag{S54}$$

In the next several sections, we use the above simplification to derive λ_{ext} for several explicit examples. See Table I for a summary.

	Dirichlet Distribution	Normalized i.i.d. Distribution	Random Delta Distribution	Nearest Neighbor Distribution
c	$\frac{\alpha}{\alpha+1}$	c	$\frac{2k+1}{4k}$	$2(1-2\mathbb{E}[\omega^2])$
D	$\frac{1}{2\alpha}\sum_{i=-k}^{k}\alpha_i i^2$	$\frac{1}{6}k(k+1)$	$\frac{1}{6}k(k+1)$	$\frac{1}{2}$
D_{ext}	$\frac{1}{2\alpha(\alpha+1)}\sum_{i=-k}^{k}\alpha_i i^2$	$\frac{1}{6}k(k+1)(1-c)$	$\frac{1}{24}(k+1)(2k-1)$	$2\mathbb{E}[\omega^2]-\frac{1}{2}$
λ_{ext}	$\frac{1}{2\alpha}$	$\frac{1-c}{2c}$	$\frac{2k-1}{2(2k+1)}$	$\frac{4\mathbb{E}[\omega^2]-1}{2(1-2\mathbb{E}[\omega^2])}$

TABLE I. A summary of the relevant coefficients for the examples in Sections VI B, VI C, VI D, and VI E.

B. Dirichlet distribution

We recall our definition of the Dirichlet distribution from Section I. The Dirichlet distribution for ν is specified by a choice of $k \in \mathbb{Z}_{\geq 1}$ and $\vec{\alpha} = (\alpha_{-k}, \ldots, \alpha_k) \in \mathbb{R}_{>0}^{2k+1}$. It assigns probability density proportional to $\prod_{j=-k}^{k} \xi(j)^{\alpha_j}$ to each vector $\xi = (\xi(-k), \ldots, \xi(k)) \in \mathbb{R}_{\geq 0}^{k}$ satisfying $\sum_{j=-k}^{k} \xi(j) = 1$. Then, we set $\xi(i) = 0$ for $i \notin [-k, k]$ so that ξ is defined on $\mathbb{Z}$.

We have

$$\mathbb{E}_\nu \left[\xi\left(i\right)\right] = \begin{cases} \frac{\alpha_i}{\alpha} & \text{for } -k \leq i \leq k \\ 0 & \text{else,} \end{cases}$$

and for $i, j \in \mathbb{Z}$, $i \neq j$:

$$\mathbb{E}_\nu \left[\xi(i)\xi(j)\right] = \frac{\alpha_i \alpha_j}{\alpha(\alpha+1)} = \frac{\alpha}{\alpha+1} \mathbb{E}_\nu \left[\xi\left(i\right)\right] \mathbb{E}_\nu \left[\xi\left(j\right)\right].$$

It follows that (S39) and (S40) hold for the Dirichlet distribution with $c = \frac{\alpha}{\alpha+1}$ so that the Dirichlet distribution falls into the general class of models considered above. Combining (10) and (S41), we see that

$$D_{\text{ext}} = \frac{1}{2\alpha(\alpha+1)} \sum_{i=-k}^{k} \alpha_i i^2.$$

We calculate the diffusion coefficient from its definition in (4),

$$D = \frac{1}{2\alpha} \sum_{i=-k}^{k} \alpha_i i^2.$$

Finally, we see from (S53) that

$$\lambda_{\text{ext}} = \frac{1}{2\alpha}. \tag{S55}$$

C. Independent and Identically Distributed Random Variables

We recall our definition of the normalized i.i.d. distribution from Section I. The normalized i.i.d. distribution for ν is specified by a choice of $k \in \mathbb{Z}_{\geq 1}$ and a choice of probability distribution on $\mathbb{R}_{\geq 0}$. Let $X_{-k}, \ldots, X_k$ be chosen independently according to the specified probability distribution, and then define $\xi(i) = X_i(\sum_{j=-k}^{k} X_j)^{-1}$ for $i \in [-k, k]$ and $\xi(i) = 0$ otherwise.

We have

$$\mathbb{E}_\nu\left[\xi\left(i\right)\right] = \begin{cases} \frac{1}{2k+1} & \text{for } -k \leq i \leq k \\ 0 & \text{else,} \end{cases}$$

and for $i, j \in \mathbb{Z}$, $i \neq j$:

$$\mathbb{E}_\nu\left[\xi(i)\xi(j)\right] = \frac{c}{(2k+1)^2}$$

for some constant $c \in (0, 1)$. Therefore, (S39) and (S40) are satisfied. Combining (10) and (S41), we see that

$$D_{\text{ext}} = \frac{1-c}{2(2k+1)} \sum_{i=-k}^{k} i^2 = \frac{1}{6} k(k+1)(1-c)$$

We calculate the diffusion coefficient from its definition in (4),

$$D = \frac{1}{2(2k+1)} \sum_{i=-k}^{k} i^2 = \frac{1}{6} k(k+1).$$

Finally, we see from (S53) that

$$\lambda_{\text{ext}} = \frac{1-c}{2c}. \tag{S56}$$

D. Random Delta Distribution

We recall our definition of the random delta distribution from Section I. The *random delta* distribution for ν is specified by $k \in \mathbb{Z}_{\geq 1}$. Two numbers X_1, X_2 are drawn uniformly without replacement from $\{-k, \ldots, k\}$. We set $\xi(X_1) = \xi(X_2) = 1/2$ and $\xi(i) = 0$ for all $i \neq X_1, X_2$.

We have

$$\mathbb{E}_\nu\left[\xi\left(i\right)\right] = \begin{cases} \frac{1}{2k+1} & \text{for } -k \leq i \leq k \\ 0 & \text{else,} \end{cases}$$

and for $i, j \in \mathbb{Z}$, $i \neq j$:

$$\mathbb{E}_\nu \left[\xi(i)\xi(j)\right] = \frac{1}{4k(2k+1)}.$$

It follows that (S39) and (S40) are satisfied with $c = \frac{2k+1}{4k}$. Combining (10) and (S41), we see that

$$D_{\text{ext}} = \frac{2k-1}{8k(2k+1)} \sum_{i=-k}^{k} i^2 = \frac{1}{24}(k+1)(2k-1).$$

We calculate the diffusion coefficient from its definition in (4),

$$D = \frac{1}{6}k(k+1).$$

Finally, we see from (S53) that

$$\lambda_{\text{ext}} = \frac{2k-1}{2(2k+1)}. \tag{S57}$$

E. Nearest Neighbor Distribution

We define the nearest neighbor distribution as follows. Let ω be any random variable taking values in the interval $[0, 1]$ such that $\mathbb{E}[\omega] = \frac{1}{2}$. We set $\xi(1) = \omega$, $\xi(-1) = 1 - \omega$ and all other $\xi(i) = 0$.

We have

$$\mathbb{E}_\nu \left[\xi\left(1\right)\right] = \mathbb{E}_\nu \left[\xi\left(-1\right)\right] = \frac{1}{2}$$

and

$$\mathbb{E}_\nu \left[\xi(1)\xi(-1)\right] = \frac{1}{2} - \mathbb{E}[\omega^2].$$

It follows that (S39) is satisfied with $c = 2(1 - 2\mathbb{E}[\omega^2])$; however, the walk $V(t)$ is no longer irreducible. In fact, when started from 0, it remains restricted to the sublattice $2\mathbb{Z}$. Therefore, $\mu(\cdot) = 0$ for all $l \notin 2\mathbb{Z}$, $\mu(0) = 1$ and $\mu(l) = c$ for $l \in 2\mathbb{Z}$. This slightly changes the analysis performed above since up until now, we were assuming that $\mu(l) = c$ for all $l \neq 0$. In particular, (S52) is replaced by

$$\sum_{l=0}^{\infty} \mathbf{E}\left[\Delta\left(t+1\right) - \Delta\left(t\right) \mid \Delta\left(t\right) = l\right] \tilde{\mu}\left(l\right) = c \sum_{l \in 2\mathbb{Z}_{\geq 0}} \sum_{|i-j|>l} (2 - \mathbb{1}_{l=0})(|i-j| - l)\mathbb{E}_\nu \left[\xi\left(i\right)\right] \mathbb{E}_\nu \left[\xi\left(j\right)\right]. \tag{S58}$$

In other words, we are only summing over even integers l. This can be further simplified in the same manner as above such that

$$\sum_{l=0}^{\infty} \mathbf{E}\left[\Delta\left(t+1\right) - \Delta\left(t\right) \mid \Delta\left(t\right) = l\right] \tilde{\mu}\left(l\right) = c \sum_{i \in \mathbb{Z}} i^2 \mathbb{E}_\nu \left[\xi\left(i\right)\right] = c.$$

Note that this is simply half of what we obtained when summing over all integers instead of just even ones. We can also see this more directly by noticing that $\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]=0$ for $l\geq 2$. Therefore,

$$\sum_{l=0}^{\infty}\mu(l)\mathbb{E}_{\nu}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]=\mathbb{E}_{\nu}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=0\right]=2(1-2\mathbb{E}[\omega^2])=c.$$

We compute $\mathrm{Var}_{\nu}\left(\mathbb{E}^{\boldsymbol{\xi}}[Y]\right)=4\mathbb{E}[\omega^2]-1=1-c$ so that

$$\lambda_{\mathrm{ext}}=\frac{\mathrm{Var}_{\nu}\left(\mathbb{E}^{\boldsymbol{\xi}}[Y]\right)}{\sum_{l=0}^{\infty}\mathbf{E}\left[\Delta\left(t+1\right)-\Delta\left(t\right)\mid\Delta\left(t\right)=l\right]\tilde{\mu}\left(l\right)}=\frac{4\mathbb{E}[\omega^2]-1}{2(1-2\mathbb{E}[\omega^2])}=\frac{1-c}{c}. \tag{S59}$$

This matches with the results in [10]. Since $D=1/2$, we have that

$$\lambda_{\mathrm{ext}}=\frac{\mathrm{Var}_{\nu}\left(\mathbb{E}^{\boldsymbol{\xi}}[Y]\right)}{\left(2D-\mathrm{Var}_{\nu}\left(\mathbb{E}^{\boldsymbol{\xi}}[Y]\right)\right)}, \tag{S60}$$

which is twice that of the irreducible cases. Similar extensions are possible if $V(t)$ is restricted to other sublattices, but we do not write out the details here.

VII. UNITS OF VARIABLES

We now introduce a lattice spacing dx and time step dt to understand the units of each variable. We do this by rescaling space and time such that $x\in dx\mathbb{Z}$ and $t\in dt\mathbb{Z}$. Note, that the environment is now defined on the new lattice meaning $\xi_{t,x}$ is a probability distribution on $dx\mathbb{Z}$. With the rescaled space and time,

$$D=\frac{1}{2dt}\sum_{i\in dx\mathbb{Z}}\mathbb{E}_{\nu}\left[\xi\left(i\right)\right]i^2 \tag{S61}$$

$$=\frac{1}{2dt}\sum_{i\in\mathbb{Z}}\mathbb{E}_{\nu}\left[\xi\left(idx\right)\right](idx)^2 \tag{S62}$$

$$=\frac{dx^2}{2dt}\sum_{i\in\mathbb{Z}}\mathbb{E}_{\nu}\left[\xi\left(idx\right)\right]i^2 \tag{S63}$$

which shows D has units area per time. The extreme diffusion coefficient, D_{ext}, has units of area per time as well since

$$D_{\mathrm{ext}}=\frac{1}{2dt}\mathbb{E}_{\nu}\left[\left(\sum_{i\in dx\mathbb{Z}}\xi_{x,t}(i)i\right)^2\right] \tag{S64}$$

$$=\frac{dx^2}{2dt}\mathbb{E}_{\nu}\left[\left(\sum_{i\in\mathbb{Z}}\xi_{x,t}(idx)i\right)^2\right]. \tag{S65}$$

We show that λ_{ext} should be a length scale via its definition in (S4). Notice the numerator of (S4) has units of area per time. The denominator has units of length per time since $\tilde{\mu}(l)$ is unitless and $\mathbf{E}[\Delta(t+dt)-\Delta(t) \mid \Delta(t)=l]$ is the change in the distance between two random walks in a single timestep. Therefore, λ_{ext} has units length. By tracking the units in our simplification of λ_{ext} in Section VI, we find the simplified form of λ_{ext} in (9) becomes

$$\lambda_{\mathrm{ext}} = \frac{1}{2}\frac{D_{\mathrm{ext}}}{(D-D_{\mathrm{ext}})}dx = \frac{1}{2}\frac{\mathrm{Var}_{\nu}\left(\mathbb{E}^{\boldsymbol{\xi}}[Y]\right)}{\mathbb{E}_{\nu}\left[\mathrm{Var}^{\boldsymbol{\xi}}(Y)\right]}dx. \tag{S66}$$

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