# Extreme Diffusion Measures Statistical Fluctuations of the Environment

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We consider many-particle diffusion in one spatial dimension modeled as "random walks in a random environment." A shared short-range space-time random environment determines the jump distributions that drive the motion of the particles. We determine universal power laws for the environment's contribution to the variance of the extreme first passage time and extreme location. We show that the prefactors rely upon a single *extreme diffusion coefficient* that is equal to the ensemble variance of the local drift imposed on particles by the random environment. This coefficient should be contrasted with the Einstein diffusion coefficient, which determines the prefactor in the power law describing the variance of a single diffusing particle and is equal to the jump variance in the ensemble averaged random environment. Thus a measurement of the behavior of extremes in many-particle diffusion yields an otherwise difficult to measure statistical property of the fluctuations of the generally hidden environment in which that diffusion occurs. We verify our theory and the universal behavior numerically over many random walk in a random environment models and system sizes.

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*Introduction*—Beneath the still surface of a glass of water lies an invisible roiling chaos; thermal motion moves the fluid environment at every time and length scale [1–10]. It is only through the motion of tracer particles that this molecular storm is rendered visible [1,2,11]. The coefficient of the mean-squared displacement power law for a single tracer particle yields a measurement of the Einstein diffusion coefficient, $D$. Classical diffusion theory [3–10] relates this to a microscopic statistic of the environment, namely the square of the mean-free path length divided by the mean collision time. This is far from a complete characterization of the thermal motion within the environment. The introduction of multiple tracer particles offers the possibility of gleaning further statistical information. However, classical diffusion theory stymies such an effort since it treats each particle as independent, thus replacing the richness and chaos of the environment with an ensemble average in which particles are randomly kicked in the same manner and at the same rate everywhere in the system. This erasure is a lie, but one which works well to predict the bulk or typical behavior of many tracer particles. In this Letter, we show that at the edges of the bulk, the truth is exposed: the fluctuations of the extremes of many tracer particles are highly sensitive to the disorder of the environment and thus reveal a more complete statistical description of the hidden environment.

Here, we study the statistical behavior of the extreme first passage time past a barrier at location $L$ and of the extreme location at time $t$ for a system of $N$ tracer particles in a general class of "random walk in random environment (RWRE) models; see Fig. 1(a). These models capture the fact that in real many-particle diffusion, all particles are subject to common and effectively random forces from the thermal fluctuations of the fluid environment. We show theoretically and confirm numerically that (1) the environment's contribution to the variances of these two observables is independent of the variance due to sampling and follows robust power laws [see (7) and (8) below] whose exponents are independent of the choice of environment, and (2) the coefficient in these power laws, as well as the time or location of onset of the power laws, depends on the Einstein diffusion coefficient, $D$, as well another parameter $D_{\mathrm{ext}}$, defined in (10), that is equal to the ensemble variance of the local drift imposed upon particles by the environment [see Fig. 1(b), where $D_{\mathrm{ext}}$ records the variance of the black arrows, and see Fig. 2 for numerical results verifying this theory]. We call $D_{\mathrm{ext}}$ the "extreme diffusion coefficient" as it relates the extreme behavior of many-particle diffusion to a microscopic statistic of the environment. In the RWRE model, the Einstein diffusion coefficient $D$ is related to a different microscopic statistic, namely the jump variance in the ensemble averaged environment. Thus, these two diffusion coefficients—that of Einstein, which is observable from a single tracer, and that introduced here, which is observable from the extremes of many tracers—offer a refined lens relative to classical diffusion theory through which to measure the statistics of the hidden environment.

*RWRE models*—In place of the independent random walk model of classical diffusion theory, we consider here lattice RWRE models. These play the role of a coarse-grained continuum environment in which particles are chaotically or thermally randomly biased in their motion

 

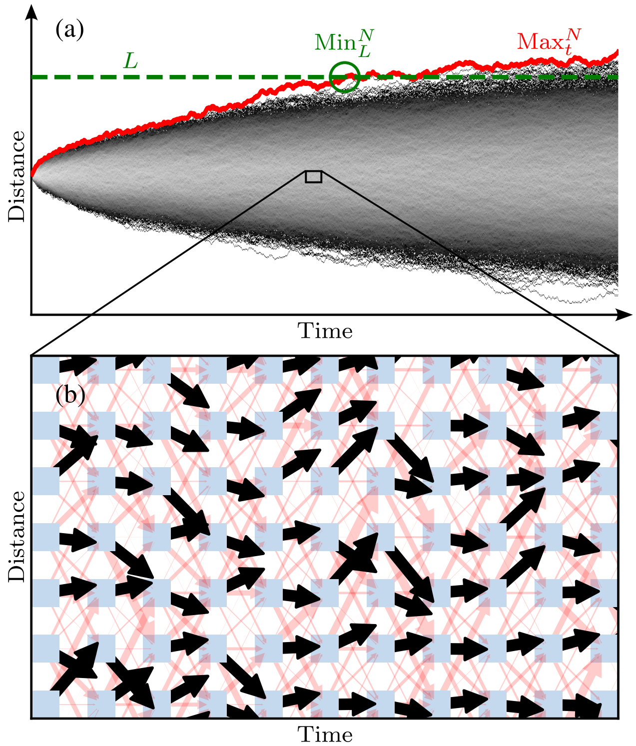


FIG. 1. (a) Space-time trajectories of $N = 10^5$ particles evolving in a random environment generated by the flat Dirichlet distribution on the interval $\{-2, \ldots, 2\}$. The solid red line denotes the extreme location $\mathrm{Max}_t^N$, and the green circle denotes the extreme first passage time $\mathrm{Min}_L^N$. (b) The random environment driving the evolution in (a): blue boxes are sites, the width of the red arrows shows the probability of jumping between sites, and black arrows are average drifts from sites.

by an environment quickly mixing in space and time. As such, we focus on RWRE models whose randomness is short-range correlated in time; see also [12–19]. This is in contrast to long-range correlated or quenched in time randomness, e.g., as in [20–30].

To define our RWRE models, we first describe how we specify the environment, and then we describe how random walks evolve therein. Each RWRE model is specified by $\nu$, a choice of probability distribution on the space of probability distributions on $\mathbb{Z}$. There are many ways to produce a random probability distribution on $\mathbb{Z}$. Perhaps the simplest involves choosing a uniform random variable on [0, 1] and then assigning that probability to $+1$ and its complementary probability to $-1$ (and 0 probability assigned to all other values of $\mathbb{Z}$). This *nearest-neighbor* model is a special case of the beta RWRE (where the uniform is replaced by a general beta distribution) introduced and studied recently in [31–36]. Though our theoretical results apply more generally, for our numerical simulations we will consider a few other specific non-nearest neighbor examples of $\nu$, namely the "Dirichlet" and "random delta" distributions described below; see also [37,39] for other examples. Let $\xi$ denote a probability distribution on $\mathbb{Z}$ sampled according to $\nu$, so that $\xi(i)$ is the probability mass on $i \in \mathbb{Z}$. For $k \geq 1$ and positive $(\alpha_{-k}, \ldots \alpha_k)$ the Dirichlet distribution is such that $\xi(i) = X_i/Z$ for $i \in [-k, k]$ [and $\xi(i) \equiv 0$ elsewhere] where the $X_i$ are independent Gamma random variables with shape $\alpha_i$ and rate $\beta = 1$ and $Z = \sum_{j \in [-k,k]} X_i$. The "flat Dirichlet" distribution is when $\alpha_i \equiv 1$ and is uniformly distributed over all probability distributions supported on $[-k, k]$. The random delta distribution is solely characterized by the width of the interval $k$. We first generate two random variables, $X_1$ and $X_2$, uniformly and without replacement on the integer interval $[-k, k]$, and then set $\xi(X_1) = \xi(X_2) = 1/2$ and $\xi(i) \equiv 0$ otherwise.

Given $\nu$, we define the random environment as $\boldsymbol{\xi} \coloneqq (\xi_{t,x} \colon x \in \mathbb{Z}, t \in \mathbb{Z}_{\geq 0})$, where each $\xi_{t,x}$ is a probability distribution on $\mathbb{Z}$ that is sampled independently according to $\nu$. The environment, $\boldsymbol{\xi}$, should be thought of as one instance of an environment in which several particles can diffuse from position $x$ and time $t$ using the transition probabilities $\xi_{t,x}$, e.g., the red arrows in Fig. 1(b).

We denote the product measure on the environment $\boldsymbol{\xi}$ by $\mathbb{P}_\nu$ and let $\mathbb{E}_\nu[\bullet]$ and $\mathrm{Var}_\nu(\bullet)$ denote the associated expectation and variance. We restrict our attention to models $\nu$ that produce net drift-free systems, i.e., such that $\sum_{j \in \mathbb{Z}} \mathbb{E}_\nu[\xi(j)]j = 0$ where $\xi$ is sampled according to $\nu$, and those with finite range, i.e., such that there exists an $M > 0$ so that $\xi(j) = 0$ if $|j| > M$. After going to a suitable moving frame, similar results to ours hold even when there is a net drift, i.e., nonzero expected mean. However, long-range, heavy-tailed jump distributions may lead to new phenomena related to anomalous diffusion (see, e.g., [30,40]), deserving further study.

Given a sample $\boldsymbol{\xi}$ of the environment, we define a probability measure $\mathbb{P}^{\boldsymbol{\xi}}$ on an arbitrary number of independent and identically distributed (i.i.d.) random walks $R^1, R^2, \ldots$ evolving in that environment, and let $\mathbb{E}^{\boldsymbol{\xi}}[\bullet]$ and $\mathrm{Var}^{\boldsymbol{\xi}}(\bullet)$ denote the associated expectation and variance. Each walk starts at $R^i(0) = 0$, where $R^i(t) \in \mathbb{Z}$ denotes its position at time $t$. The probability that a walk at $x$ at time $t$ transitions to $x + j$ at time $t + 1$ is

$$\mathbb{P}^{\boldsymbol{\xi}}(R^i(t+1) = x + j | R^i(t) = x) = \xi_{t,x}(j).$$

All random walks evolve independently given $\boldsymbol{\xi}$, though notably walkers at the same location $x$ at the same time $t$ are subject to the same jump distribution, $\xi_{t,x}$. We define the transition probabilities given $\boldsymbol{\xi}$ by $p^{\boldsymbol{\xi}}(x, t) = \mathbb{P}^{\boldsymbol{\xi}}(R(t) = x)$ [we drop the superscript $i$ here and elsewhere below since each $R^i(t)$ is i.i.d.]; this satisfies $p^{\boldsymbol{\xi}}(x, 0) = \mathbf{1}_{x=0}$, and

$$p^{\boldsymbol{\xi}}(x, t+1) = \sum_{j \in \mathbb{Z}} p^{\boldsymbol{\xi}}(x - j, t) \xi_{t,x-j}(j). \qquad (1)$$

The measure $\mathbb{P}_\nu$ is on the environment $\boldsymbol{\xi}$ and $\mathbb{P}^{\boldsymbol{\xi}}$ is on independent random walks given the environment $\boldsymbol{\xi}$. It is also useful to define $\mathbf{P}$ by $\mathbf{P}(\bullet) = \mathbb{E}_\nu[\mathbb{P}^{\boldsymbol{\xi}}(\bullet)]$ for any event $\bullet$ defined in terms of the random walks $R^1, R^2, \ldots$, and to let $\mathbf{E}[\bullet]$ and $\mathbf{Var}[\bullet]$ denote the associated expectation and variance. This is the marginal distribution on many-particle diffusion trajectories that is relevant to repeated experimental studies of many-particle diffusion; $\mathbf{P}$ represents the histogram over many experimental samples of the many-particle diffusion trajectories.

Given a random walk $R^i$, we denote its first passage time past $L > 0$ by $\tau_L^i$. The "extreme first passage time" of $N$ random walkers past location $L$ is defined as

$$\mathrm{Min}_L^N := \min\left(\{\tau_L^1, \ldots, \tau_L^N\}\right).$$

Given an environment $\boldsymbol{\xi}$, the walks $R^1, \ldots, R^N$ are i.i.d. under the measure $\mathbb{P}^{\boldsymbol{\xi}}$, and thus so are the $\tau_L^i$. Therefore,

$$\mathbb{P}^{\boldsymbol{\xi}}\left(\mathrm{Min}_L^N \le t\right) = 1 - \left(1 - \mathbb{P}^{\boldsymbol{\xi}}(\tau_L \le t)\right)^N. \tag{2}$$

We also study the *extreme location* of $N$ walks at time $t$,

$$\mathrm{Max}_t^N := \max\left(\{R^1(t), \ldots R^N(t)\}\right).$$

Given $\boldsymbol{\xi}$, under the measure $\mathbb{P}^{\boldsymbol{\xi}}$ on $R^1, \ldots, R^N$,

$$\mathbb{P}^{\boldsymbol{\xi}}\left(\mathrm{Max}_t^N \ge x\right) = 1 - \left(1 - \mathbb{P}^{\boldsymbol{\xi}}(R(t) \ge x)\right)^N. \tag{3}$$

We characterize the statistics of $\mathrm{Min}_L^N$ and $\mathrm{Max}_t^N$ under the measure $\mathbf{P}$, i.e., when the environment is hidden as in real diffusive systems. There are two levels of randomness: first, the randomness due to the environment, $\mathrm{Env}_t^N$, and second, the randomness of sampling walkers in that environment, $\mathrm{Sam}_L^N$. Specifically, we define $\mathrm{Env}_L^N$ as the minimum time $t$ such that $\mathbb{P}^{\boldsymbol{\xi}}(\tau_L \le t) \ge (1/N)$ and $\mathrm{Sam}_L^N$ as the residual, $\mathrm{Sam}_L^N := \mathrm{Min}_L^N - \mathrm{Env}_L^N$. Note that $\mathrm{Env}_L^N$ only depends on the environment $\boldsymbol{\xi}$ and by (2), $\mathrm{Env}_L^N$ satisfies $\mathbb{P}^{\boldsymbol{\xi}}(\mathrm{Min}_L^N \le \mathrm{Env}_L^N) \approx 1 - e^{-1}$. Thus, $\mathrm{Env}_L^N$ is approximately the mean, or median, of $\mathrm{Min}_L^N$ in a given environment and $\mathrm{Sam}_L^N$ captures fluctuations about $\mathrm{Env}_L^N$ due to sampling $N$ random walks in that environment. If one were to sample many systems of $N$ independent random walks in the *same* environment, one could compute the mean value of $\mathrm{Min}_L^N$ (i.e., the mean extreme first passage time) and find that it is well approximated by $\mathrm{Env}_L^N$. The fluctuations of the many different measured values of $\mathrm{Min}_L^N$, in the same environment, will in turn be captured by $\mathrm{Sam}_L^N$. We similarly define $\mathrm{Env}_t^N$ as the maximum position $x$ satisfying $\mathbb{P}^{\boldsymbol{\xi}}(R(t) \ge x) \ge (1/N)$ and $\mathrm{Sam}_t^N := \mathrm{Max}_t^N - \mathrm{Env}_t^N$. Note the subscript of $\mathrm{Env}_L^N$, $\mathrm{Env}_t^N$, $\mathrm{Sam}_L^N$, and $\mathrm{Sam}_t^N$ distinguishes between measurements of the first passage at a distance $L$ (subscript $L$) and location of extreme particle at time $t$ (subscript $t$).

*Theoretical results*—The Einstein diffusion coefficient,

$$D := \frac{1}{2}\sum_{j\in\mathbb{Z}} \mathbb{E}_\nu[\xi(j)]j^2, \tag{4}$$

for the RWRE is defined as the variance of the ensemble averaged jump distribution (recall that we have assumed a net drift-free system, i.e., $\sum_{j\in\mathbb{Z}} \mathbb{E}_\nu[\xi(j)]j = 0$). The following results are derived under the assumption that $L \gg \lambda_{\mathrm{ext}} \ln(N)^{3/2}$ or $t \gg (\lambda_{\mathrm{ext}}^2/D)\ln(N)^2$ [where $\lambda_{\mathrm{ext}}$ is defined in (10)], and $N \gg 1$. We find that

$$\mathbf{E}\left[\mathrm{Min}_L^N\right] \approx \frac{L^2}{4D\ln(N)}, \qquad \mathbf{E}\left[\mathrm{Max}_t^N\right] \approx \sqrt{4D\ln(N)t}$$

matching the classical theory of diffusion, i.e., where the RWRE is replaced by independent random walks with Einstein diffusion coefficient $D$; see, e.g., [41–47].

The variance of $\mathrm{Min}_L^N$ and $\mathrm{Max}_t^N$ reveals more interesting behavior. The environmental and sampling contributions to $\mathrm{Min}_L^N$ and $\mathrm{Max}_t^N$ are roughly independent,

$$\mathbf{Var}(\mathrm{Min}_L^N) \approx \mathbf{Var}(\mathrm{Env}_L^N) + \mathbf{Var}(\mathrm{Sam}_L^N), \tag{5}$$

$$\mathbf{Var}(\mathrm{Max}_t^N) \approx \mathbf{Var}(\mathrm{Env}_t^N) + \mathbf{Var}(\mathrm{Sam}_t^N). \tag{6}$$

The sampling fluctuations are centered Gumbel with

$$\mathbf{Var}(\mathrm{Sam}_L^N) \approx \frac{\pi^2 L^4}{96D^2\ln(N)^4}, \qquad \mathbf{Var}(\mathrm{Sam}_t^N) \approx \frac{\pi^2 Dt}{6\ln(N)},$$

in agreement with classical diffusion theory [41–47]. The environmental fluctuations follow anomalous power laws

$$\mathbf{Var}(\mathrm{Env}_L^N) \approx \lambda_{\mathrm{ext}} \frac{\sqrt{2\pi}L^3}{8D^2\ln(N)^{5/2}} \tag{7}$$

$$\mathbf{Var}(\mathrm{Env}_t^N) \approx \lambda_{\mathrm{ext}}\sqrt{2\pi Dt}. \tag{8}$$

where $D$ is the Einstein diffusion coefficient,

$$\lambda_{\mathrm{ext}} := \frac{1}{2}\frac{D_{\mathrm{ext}}}{(D - D_{\mathrm{ext}})} = \frac{1}{2}\frac{\mathrm{Var}_\nu(\mathbb{E}^{\boldsymbol{\xi}}[Y])}{\mathbb{E}_\nu[\mathrm{Var}^{\boldsymbol{\xi}}(Y)]}, \tag{9}$$

and $D_{\mathrm{ext}}$ is the extreme diffusion coefficient,

$$D_{\mathrm{ext}} := \frac{1}{2}\mathrm{Var}_\nu(\mathbb{E}^{\boldsymbol{\xi}}[Y]) = \frac{1}{2}\mathbb{E}_\nu\left[\left(\sum_{j\in\mathbb{Z}}\xi(j)j\right)^2\right]. \tag{10}$$

See the Appendix' for a brief discussion of our theoretical methods and [37] for a derivation of our results.

Here and below, $Y$ is a random jump distributed according to $\xi$. Thus $D_{\mathrm{ext}}$ is the variance over the random environment of the drift of a single jump, i.e., how much the black arrows fluctuate over space and time in Fig. 1(b). Then $\lambda_{\rm ext}$ is the ratio of that to the mean over the random environment of the variance of a single jump. The formula in (9) is derived in [37] under the assumptions

$$\mathbb{E}_\nu[\xi(i)\xi(j)] = c\mathbb{E}_\nu[\xi(i)]\mathbb{E}_\nu[\xi(j)] \quad \text{for all } i \neq j, \tag{11}$$

with some fixed $c \in (0,1)$, and the RWRE is aperiodic (i.e., it does not live on a strict space-time sublattice). In terms of $c$, $\lambda_{\rm ext} = ((1-c)/2c)$. When (11) fails, there is a more involved formula for $\lambda_{\rm ext}$; see Refs. [37,39,48]. When aperiodicity fails, e.g. for the nearest-neighbor RWRE, an additional factor arises in (9); see Ref. [37].

The extreme diffusion coefficient necessarily satisfies $D_{\rm ext} \in [0, D]$. When $D_{\rm ext} = D$ (hence $\lambda_{\rm ext} = \infty$), $\xi$ is supported at a single, yet random site, hence a perfectly sticky environment of coalescing random walkers. There is a scaling limit where $D_{\rm ext} \to D$ as time is rescaled, which leads to sticky Brownian motions as studied, for instance, in [33,39,49,50]. When $D_{\rm ext} = 0$, (hence $\lambda_{\rm ext} = 0$), the drift $\mathbb{E}^\xi[Y]$ becomes deterministic, thus with probability 1 under $\mathbb{P}_\nu$, $\sum_j \xi(j)j$ is constant, and (by the net drift-free assumption) equal to 0; see Refs. [39,51].

The quantities $D$ and $D_{\rm ext}$ have units of area per time, while $\lambda_{\rm ext}$ has units of length. This is shown by rescaling the lattice so as to endow it with length and time units (see Ref. [37]) or by studying certain continuum versions of our model such as sticky Brownian motions [33,39,49,50] or Langevin dynamics in a random environment [32,52].

*Numerical results*—Figures 2(a) and 2(e) show that our theoretical prediction for each relevant variance is asymptotically accurate and that the addition laws in (5) and (6) are justified. Figures 2(b) and 2(f) show the collapse of the scaled environmental variance for several choices of $\nu$. Although our predictions assume $N$ is large, they are accurate for systems as small as $N = 100$. This is also shown in Figs. 2(d) and 2(h) since the ratio of the measured environmental variance to the theoretical prediction goes

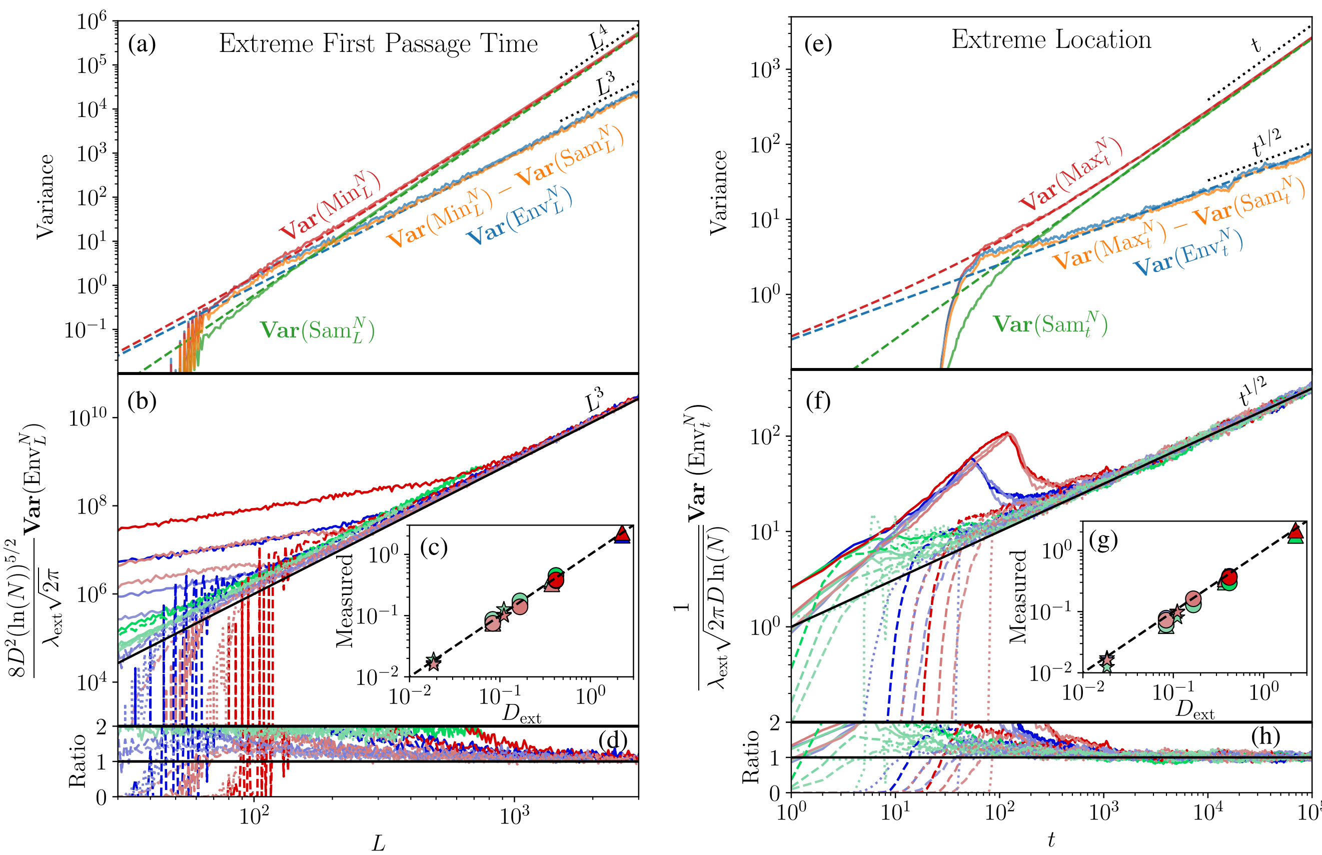


FIG. 2. Plot of the measured variances for the extreme first passage time (a) and extreme location (e) for systems of $N = 10^{28}$ and the flat Dirichlet distribution with $D = 1$. (b),(f) Collapse of the environmental variance as a function of $L$ and $t$, respectively, $N = 10^2, 10^{14}$, and $10^{28}$ is plotted in red, blue, and green, respectively; the Dirichlet, flat Dirichlet, and random delta distributions for $\nu$ are rendered as dotted, dashed, and solid lines, respectively; the color saturation is proportional to the Einstein diffusion coefficient $D$. Insets (c),(g): plot of the measured $D_{\rm ext}$ against the true value of $D_{\rm ext}$. The dashed line represents equality. The Dirichlet, flat Dirichlet, and random delta distributions are labeled with a star, circle, and triangle, respectively. (d),(h) Plot of ratios of measured environmental variance vs theoretical prediction in (7) and (8) as a function of $L$ and $t$, respectively. All reported quantities are measured by simulating 500 different systems for a given $\nu$ and $N$.

to 1, asymptotically. Figures 2(c) and 2(g) show that our numerically computed values of $D_{\text{ext}}$ from both kinds of measurements match the theoretical value, falling onto the line of equality. Since $\mathbf{Var}(\text{Sam}_L^N)$, and similarly $\mathbf{Var}(\text{Sam}_t^N)$, rely only on $D$, a measurement of $\text{Min}_L^N$, or $\text{Max}_t^N$ can be translated to a measurement of $D_{\text{ext}}$. See the Appendix for our numerical methods.

*Conclusion*—While we have focused herein on lattice models, the theory of extreme diffusion should extend to systems with continuous space and time dimensions provided sufficiently fast mixing of the random environment in both dimensions. Besides demonstrating a universality result, the key contribution of this Letter is the link shown between the extreme diffusion coefficient and the statistics of the environment. Previous works on extreme diffusion have considered nearest-neighbor RWREs [31–36], which are too simple to reveal this relationship, or the continuum sticky Brownian motions where there is no clear notion of an environment [33,39,49,50]). In light of the universality we demonstrate here, we expect this model to be applicable to all physical systems in which particles (or more generally, agents) move in response to the forcing present within a shared space-time chaotic environment. There are several outstanding theoretical challenges, including (1) establishing the role of correlation length and timescales in defining the continuum extreme diffusion coefficient, (2) extending extreme diffusion theory to higher spatial dimensions, (3) understanding whether there exists a fluctuation-dissipation type relation for the extreme diffusion coefficient, and if so, what the relevant notion of extreme drag is, and (4) testing extreme diffusion in experimental systems.

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## End Matter

*Appendix—*

Theoretical methods: Our derivation [37] of the extreme first passage and location statistics closely follows [34,35] and relies on moderate deviation asymptotics for the tail probability $\mathbb{P}^{\boldsymbol{\xi}}(R(t) \geq x)$ with $x \propto t^{3/4}$, where $\boldsymbol{\xi}$ is distributed according to $\mathbb{P}_{\nu}$. We rely on results from [39,48] that prove convergence in this regime of the tail probability to the Kardar-Parisi-Zhang (KPZ) equation [53,54].

The occurrence of the KPZ equation under moderate deviation scaling was predicted (without a formula for the KPZ coefficients) first in [32] and confirmed with coefficients for the beta RWRE in [33] and general nearest-neighbor RWRE in [36], all prior to the general case addressed in [39,48].

To see this convergence, first observe that the master equation (1) implies the tail probability $\mathbb{P}^{\boldsymbol{\xi}}[R(t) \geq x]$ is the partition function for a directed polymer model. The polymer replica method, developed first in the continuum setting (and related to the KPZ equation) [55,56], expresses the $k$th moment of the tail probability as the expected value over $k$ independent random walks of a certain interaction energy. In the continuum limit, this interaction energy becomes equal to the coefficient $\lambda_{\text{ext}}$ times the sum of the pair local times (i.e., coming from a Dirac delta interaction potential) of $k$ Brownian motions, and thus recovering the $k$th moment of the continuum directed polymer, i.e., the multiplicative stochastic heat equation. The coefficient $\lambda_{\text{ext}}$ can already be determined from the $k = 2$ case and involves the two-point motion, i.e., the law of $R^1$ and $R^2$ under $\mathbf{P}$, via the expected change in its difference $|R^1(t) - R^2(t)|$ in a unit of time when initialized under its invariant measure. This is derived in [48] (see also [39]) using a discrete Tanaka formula to relate the discrete and continuum interaction potential. Under (11), the invariant measure for the two-point motion becomes constant away from the origin and $\lambda_{\text{ext}}$ simplifies to yield (9).

As in [34,35], the tail probability $\mathbb{P}^{\boldsymbol{\xi}}(R(t) \geq x)$ can be inverted to approximately recover $\text{Env}_L^N$ and $\text{Env}_t^N$. By the KPZ convergence, this readily implies that

$$\mathbf{Var}(\text{Env}_L^N) \approx \frac{L^4}{(4D)^2 \ln(N)^4} \text{Var}\left(h\left(\frac{32\lambda_{\text{ext}}^2 \ln(N)^3}{L^2}, 0\right)\right),$$
$$\mathbf{Var}(\text{Env}_t^N) \approx \frac{Dt}{\ln(N)} \text{Var}\left(h\left(\frac{8\lambda_{\text{ext}}^2 \ln(N)^2}{Dt}, 0\right)\right),$$

where $h(t,x)$ is the narrow-wedge solution to the KPZ equation. The power laws and their regime of applicability come from the short-time Edwards-Wilkinson asymptotics of the KPZ equation; see e.g., [57,58].

Numerical methods: We describe how we numerically measure the mean and variance of $\text{Env}_L^N$, $\text{Sam}_L^N$, and $\text{Min}_L^N$. We begin by numerically computing the probability mass function of the first passage time for a single particle [we drop the superscript and just call it $R(t)$], defined as $\tau_L = \min(t : R(t) \geq L)$. To do so, we consider $R_L(t) = R(\min(t, \tau_L))$, the random walk stopped (or absorbed) when $R(t) \geq L$. We denote the probability mass function of $R_L(t)$ as $p_L^{\boldsymbol{\xi}}(x,t) = \mathbb{P}^{\boldsymbol{\xi}}(R_L(t) = x)$. Given an environment $\boldsymbol{\xi}$, $p_L^{\boldsymbol{\xi}}(x,t)$ uniquely solves

$$p_L^{\boldsymbol{\xi}}(x, t+1) = \sum_{i<L} p_L^{\boldsymbol{\xi}}(i,t)\xi_{t,i}(x-i) \tag{A1}$$

for $x \in (-\infty, L) \cap \mathbb{Z}$ and $t \geq 0$, subject to the absorbing boundary condition

$$p_L^{\boldsymbol{\xi}}(L, t+1) = p_L^{\boldsymbol{\xi}}(L,t) + \sum_{i<L} p_L^{\boldsymbol{\xi}}(i,t) \sum_{j \geq L} \xi_{t,i}(j-i) \tag{A2}$$

and initial condition $p_L^{\boldsymbol{\xi}}(x,t) = \mathbb{1}_{x=0}$. The probability of $\tau_L$ occurring before time $t$ is given by the probability of $R_L(t)$ being absorbed before time $t$, which is to say,

$$\mathbb{P}^{\boldsymbol{\xi}}(\tau_L \leq t) = p_L^{\boldsymbol{\xi}}(L,t). \tag{A3}$$

For a given environment distribution $\nu$, we numerically sample $\nu$ to generate an environment, $\boldsymbol{\xi}$, and then we use this environment to compute $\mathbb{P}^{\boldsymbol{\xi}}(\tau_L \leq t)$ via (A1) and (A2). We measure $\text{Env}_L^N$ in this environment by finding the minimum time $t$ such that $\mathbb{P}^{\boldsymbol{\xi}}(\tau_L \leq t) \geq 1/N$. We then numerically compute the distributions of $\text{Min}_L^N$ and $\text{Sam}_L^N$ in the given environment. For $\text{Min}_L^N$ we use (2) and for $\text{Sam}_L^N$ we combine (2) and our definition $\text{Sam}_L^N \coloneqq \text{Min}_L^N - \text{Env}_L^N$ and thus compute the distribution of $\text{Sam}_L^N$ as

$$\mathbb{P}^{\boldsymbol{\xi}}(\text{Sam}_L^N \leq t) = 1 - \left(1 - \mathbb{P}^{\boldsymbol{\xi}}\left(\tau_L \leq t + \text{Env}_L^N\right)\right)^N. \tag{A4}$$

Since $N$ is quite large, we use arbitrary precision floating point arithmetic to compute the $N$th power in (2) and (A4). Now, given these computed distribution functions we compute $\mathbb{E}^{\boldsymbol{\xi}}[\text{Min}_L^N]$ and $\text{Var}^{\boldsymbol{\xi}}(\text{Min}_L^N)$, $\mathbb{E}^{\boldsymbol{\xi}}[\text{Sam}_L^N]$ and $\text{Var}^{\boldsymbol{\xi}}(\text{Sam}_L^N)$. Finally, we repeat this procedure for many different samples of $\boldsymbol{\xi}$. We compute $\mathbf{E}[\text{Env}_L^N]$ and $\mathbf{Var}(\text{Env}_L^N)$ by taking the mean and variance of $\text{Env}_L^N$ over these samples. For $\text{Min}_L^N$ and $\text{Sam}_L^N$ we utilize the law of total expectation, i.e., $\mathbf{E}[\text{Min}_L^N] = \mathbb{E}_\nu[\mathbb{E}^{\boldsymbol{\xi}}[\text{Min}_L^N]]$ (likewise for $\text{Sam}_L^N$) and the total law of variance $\mathbf{Var}(\text{Min}_L^N) = \text{Var}_\nu(\mathbb{E}^{\boldsymbol{\xi}}[\text{Min}_L^N]) + \mathbb{E}_\nu[\text{Var}^{\boldsymbol{\xi}}(\text{Min}_L^N)]$.

We use a nearly identical procedure to numerically compute the mean and variance of the corresponding extreme location quantities $\text{Env}_t^N$, $\text{Sam}_t^N$, and $\text{Max}_t^N$. Given an environment $\boldsymbol{\xi}$ we start by numerically computing $p^{\boldsymbol{\xi}}(x,t)$ using the recurrence relation given in (1). We then calculate $\text{Env}_t^N$ by finding the maximum position $x$ such that $\mathbb{P}^{\boldsymbol{\xi}}(R(t) \geq x) \geq 1/N$. We compute the distribution of $\text{Max}_t^N$ using (3), and the distribution of $\text{Sam}_t^N$ by combining (3) and our definition $\text{Sam}_t^N \coloneqq \text{Max}_t^N - \text{Env}_t^N$ to find

$$\mathbb{P}^{\boldsymbol{\xi}}(\text{Sam}_t^N \leq x) = (1 - \mathbb{P}^{\boldsymbol{\xi}}(R(t) \leq x + \text{Env}_t^N))^N. \tag{A5}$$

From these we compute $\mathbb{E}^{\boldsymbol{\xi}}[\text{Max}_t^N]$, $\text{Var}^{\boldsymbol{\xi}}(\text{Max}_t^N)$, $\mathbb{E}^{\boldsymbol{\xi}}[\text{Sam}_t^N]$, and $\text{Var}^{\boldsymbol{\xi}}(\text{Sam}_t^N)$. Finally, by repeating for several samples of $\boldsymbol{\xi}$, as above, we compute $\mathbf{E}[\text{Env}_t^N]$ and $\mathbf{Var}(\text{Env}_t^N)$, and then $\mathbf{E}[\text{Max}_t^N]$, $\mathbf{E}[\text{Sam}_t^N]$, $\mathbf{Var}(\text{Max}_t^N)$, and $\mathbf{Var}(\text{Sam}_t^N)$.

One could measure $\mathrm{Min}_L^N$ and $\mathrm{Max}_t^N$ using agent based simulations in contrast to generating their probability distributions as we do here. By generating the probability distributions, we are also able to measure the environmental and sampling fluctuations.

The extreme diffusion coefficient, $D_{\mathrm{ext}}$, can be independently measured from both the extreme first passage and extreme location statistics. Using (9) we find $D_{\mathrm{ext}} = (2\lambda_{\mathrm{ext}} D/(1+2\lambda_{\mathrm{ext}}))$ [$D$ is computed using (4) though it could also be recovered numerically]. For the extreme first passage time, $\lambda_{\mathrm{ext}}$ is computed as $(8D^2 \ln(N)^{5/2}/\sqrt{2\pi}L^3) \times (\mathbf{Var}(\mathrm{Min}_L^N) - \mathbf{Var}(\mathrm{Sam}_L^N))$, whereas for the extreme location, $\lambda_{\mathrm{ext}}$ is computed as $(1/\sqrt{2\pi D t})(\mathbf{Var}(\mathrm{Max}_t^N) - \mathbf{Var}(\mathrm{Sam}_t^N))$.

We simulate a Dirichlet distribution with $\vec{\alpha} = (12, 1, 12)$, which is peaked at $-1$ and 1 with particles having a small probability of staying at the same location. We also simulate a Dirichlet distribution with $\vec{\alpha} = (2, 1, 1/4, 4, 1/2)$ to study a distribution that is not symmetric in the average environment. We consider the flat Dirichlet distribution on the interval $[-k, k]$ for $k = 1$, 2, 5. Recall that the flat Dirichlet distribution is a special case of the Dirichlet distribution with all $\alpha_i = 1$. Lastly, we consider the random delta distribution on the interval $[-k, k]$ for $k = 1$, 2, 5.